

Lec 10

Key Properties \mathbb{R}^n



any vector space V

- span

- subspace

- linear dependence.

- basis

- linear transformation: $V \rightarrow W$

- coordinate.

...

Ex. $V = \mathbb{R}^{m \times n}$

addition $A+B$



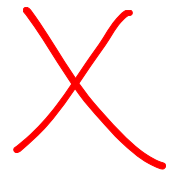
scalar mult cA .

Ex. $V = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

addition : $x+y := xy$

mult : $cx := cx$.

not closed under scalar mult.



$$\underline{\Sigma x}. \quad V = \left\{ f(x) = \sum_{n=1}^N a_n \cos(nx) \mid a_n \in \mathbb{R}, x \in \mathbb{R} \right\}$$

$$\text{add: } (f+g)(x) := f(x) + g(x)$$

$$\text{mult: } (c f)(x) = c \cdot f(x) \quad \checkmark$$

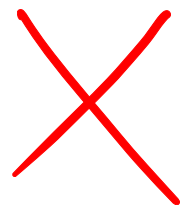
$$\underline{\Sigma x}. \quad V = \mathbb{Z} = \{ \text{integers} \}.$$

$$\text{add: } x + y$$

$$\text{mult: } cx.$$

$$c = \frac{1}{2}, \quad x = 1,$$

$$cx = \frac{1}{2} \notin \mathbb{Z}$$



Ex. $V = \mathbb{Q} = \{ \text{rational number.} \}$

$$\frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0 \}$$

$$c = \pi$$

$$x = 1.$$

$$c \cdot x = \pi \notin \mathbb{Q}$$

X

Σx . $V = \mathbb{P}_3$

Find a basis of V .

Guess: $\{1, x, x^2, x^3\}$. $\dim V = 4$

① (not too big, i.e., lin. indep.)

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0.$$

Fundamental thm of algebra.

e.g. $x^2 - 1 = 0$. roots $x_1 = 1, x_2 = -1$

$x^2 + 1 = 0$. roots $x_1 = i, x_2 = -i$

$$i^2 = -1$$

$x^2 = 0$, roots $x = 0$ w. multiplicity
2.

$a_0 + a_1x + a_2x^2 + a_3x^3 = 0$ has at most

3 roots in \mathbb{C} . unless $a_0 = a_1 = a_2 = a_3 = 0$

\Rightarrow trivial sol. only.

\Rightarrow lin. indep.

② (not too small, i.e. $\text{span} = V$)

by definition.

\square .

\mathbb{P}_n basis $\{1, x, x^2, \dots, x^n\}$.

$$\dim \mathbb{P}_n = n+1.$$

$\mathbb{P} = \{\text{polynomials of finite degree}\}$.

$$\dim \mathbb{P} = \infty.$$

RARELY USED in THIS CLASS.

Ex . $V = W = \mathbb{P}_3$

$$T: V \rightarrow W$$

$$f \mapsto \frac{df}{dx}$$

① Is T a lin. trans.?

② Image(T) . Null(T)

③ Basis



Step 0 . SHOW $T : P_3 \rightarrow P_3$
is well defined.

Take any $f \in P_3$,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

$$(Tf)(x) = \frac{df}{dx}(x) = a_1 + 2a_2x + 3a_3x^2.$$

$$\in P_2 \subseteq P_3.$$

① Linearity.

$$\begin{aligned} T(f+g) &= \frac{d}{dx}(f+g) \\ &= \frac{df}{dx} + \frac{dg}{dx} = T(f) + T(g) \end{aligned}$$

$$\begin{aligned} T(cf) &= \frac{d}{dx}(cf) \\ &= c \frac{df}{dx}. \end{aligned}$$

Therefore T is a well defined
lin. trans.

$$\textcircled{2} \text{ Image}(T) = \{ Tf \mid f \in P_3 \} = P_2$$

$$\begin{aligned} \text{Null}(T) &= \{ f \in P_3 \mid Tf = 0 \} \\ &= \{ a_0 \mid a_0 \in \mathbb{R} \} = P_0 \end{aligned}$$

③ basis for $\text{Image}(T)$ $\{1, x, x^2\}$.

" " $\text{Null}(T)$ $\{1\}$

