Lec 10

Key Properties $\mathbb{R}^n$ → any vector space V

- span
- subspace
- linear dependence.
- basis
- linear transformation : $V \to W$

- coordinate.
- ...

Ex. $V = \mathbb{R}^{m \times n}$

addition $A+B$ ✓

scalar mult cA.

Ex. $V = \mathbb{R}^{+} = \{x \in \mathbb{R} \mid x > 0\}$.

addition : $x + y := xy$

mult : $c x := cx$.

not closed under scalar mult. ✗

<u>Ex</u>. $V = \{ f(x) = \sum_{n=1}^{N} a_n \cos(nx) \mid a_n \in \mathbb{R}, x \in \mathbb{R} \}$

add: $(f+g)(x) := f(x) + g(x)$

mult: $(cf)(x) = c \cdot f(x)$ ✓

<u>Ex</u>. $V = \mathbb{Z} = \{\text{integers}\}$.

add: $x + y$

mult: cx.

$c = \frac{1}{2}$, $x = 1$,

$cx = \frac{1}{2} \notin \mathbb{Z}$ ✗

Ex. $V = \mathbb{Q} = \{$ rational number $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0\}$.

$c = \pi$ $x = 1$. $c \cdot x = \pi \notin \mathbb{Q}$ $\times$

Ex. $V = \mathbb{P}_3$

Find a basis of V.

Guess: $\{1, x, x^2, x^3\}$. dim $V = 4$

① (not too big, i.e., lin. indep.)

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0.$$

Fundamental thm of algebra.

e.g. $x^2-1=0$. roots $x_1=1, x_2=-1$

$x^2+1=0$. roots $x_1=i, x_2=-i$

$i^2=-1$

$x^2=0$. roots $x=0$ w. multiplicity 2.

$a_0+a_1x+a_2x^2+a_3x^3=0$ has at most

3 roots in $\mathbb{C}$. unless $a_0 = a_1 = a_2 = a_3 = 0$

$\Rightarrow$ trivial sol. only.

$\Rightarrow$ lin. indep.

② (not too small, i.e. span $= V$)

by definition.

$\square$.

$\mathbb{P}_n$ basis $\{1, x, x^2, \ldots, x^n\}$.

$\dim \mathbb{P}_n = n+1$.

$\mathbb{P} = \{\text{polynomials of finite degree}\}$.

$\dim \mathbb{P} = \infty$.

RARELY USED in THIS CLASS.

Ex. $V = W = \mathbb{P}_3$

$$T : V \longrightarrow W$$

$$f \longmapsto \frac{df}{dx}$$

① Is T a lin. trans?

② Image(T). Null(T)

③ Basis

Step 0. SHOW $T : \mathbb{P}_3 \to \mathbb{P}_3$ is well defined.

Take any $f \in \mathbb{P}_3$,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

$$(Tf)(x) = \frac{df}{dx}(x) = a_1 + 2a_2 x + 3a_3 x^2.$$

$$\in \mathbb{P}_2 \subseteq \mathbb{P}_3.$$

① Linearity.

$$T(f+g) = \frac{d}{dx}(f+g)$$

$$= \frac{df}{dx} + \frac{dg}{dx} = T(f) + T(g)$$

$$T(cf) = \frac{d}{dx}(cf)$$

$$= c\frac{df}{dx}.$$

Therefore T is a well defined lin. trans.

② $\text{Image}(T) = \{ Tf \mid f \in \mathbb{P}_3 \} = \mathbb{P}_2$

$\text{Null}(T) = \{ f \in \mathbb{P}_3 \mid Tf = 0 \}$

$= \{ a_0 \mid a_0 \in \mathbb{R} \} = \mathbb{P}_0$

③ basis for Image(T) $\{1, x, x^2\}$.

" " Null(T) $\{ 1 \}$