Lec 11.

Ex. Is $f(x)=1+x-x^3$ in $\mathrm{span}\{1+x^3, 1-2x, x^2-x^3\}$?

If $f(x)$ is in the span, then $\exists\ a_1, a_2, a_3$ s.t.

$$f(x) = a_1(1+x^3) + a_2(1-2x) + a_3(x^2-x^3)$$

$$= (a_1+a_2) - 2a_2 x + a_3 x^2 + (a_1-a_3)x^3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

lin. sys.

$$\xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-\frac{5}{2}} \end{array}\right]$$

pivot

no sol $\Rightarrow$ NOT in span.

<u>Ex</u>. $V = \mathbb{P}_2$. Find ALL vectors in V of the form

$$f(x) = a_0 + a_1 x + a_2 x^2$$

s.t. $\{1, x, f(x)\}$ forms a basis of V.

① $\{1, x, f(x)\}$ lin. indep.

② $\operatorname{span}\{1, x, f(x)\} = V = \mathbb{P}_2$.

$$[\underset{\vec{a}_1}{\underset{\uparrow}{1}},\ \underset{\vec{a}_2}{\underset{\uparrow}{x}},\ \underset{\vec{a}_3}{\underset{\uparrow}{f(x)}}] = [1, x, x^2]\begin{bmatrix} 1 & 0 & a_0 \\ 0 & 1 & a_1 \\ 0 & 0 & \boxed{a_2} \end{bmatrix}$$

pivot.

$a_2 \neq 0$.

Ex. $V = \mathbb{R}^{2\times 2}$.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Is $\{A_1, A_2, A_3, A_4\}$ lin. indep?

$$x_1 A_1 + x_2 A_2 + x_3 A_3 + x_4 A_4 = 0$$

has only trivial sol?

basis for V.

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_1 = \underbrace{[e_1, e_2, e_3, e_4]}_{\text{basis}} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}_{\text{coordinate}}.$$

$$[A_1 \ A_2 \ A_3 \ A_4] = [e_1 \ e_2 \ e_3 \ e_4] \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Column vectors of coordinate matrix are lin. indep?

$\xrightarrow{REF}$ $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ → no pivot.

→ free variable.

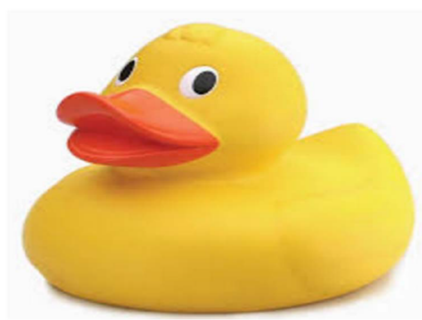
→ lin. dep.

To go beyond 2d/3d Euclidean space

Conceptualize

Ex. $V =$ collection of all possible ducks differing only in width

$\vec{v}_1$



$\vec{v}_2$

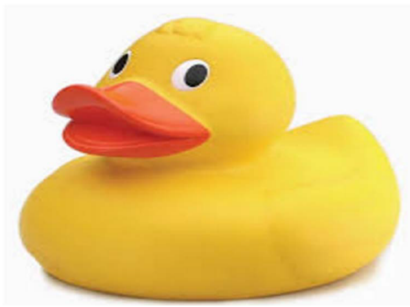
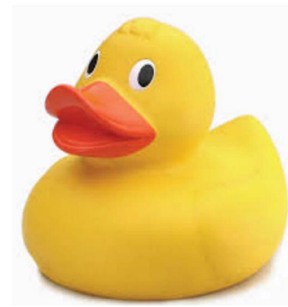
$\vec{v}_3$

...

Pick one duck $\vec{b} \in \vec{V}$, $\vec{b} \neq \vec{0}$

$\vec{b}$

For all other $\vec{v} \in V$, $\vec{v} = a\vec{b}$, $a \in \mathbb{R}$

 $=$ 1.8 

$$\vec{v} = \underbrace{1.8}_{\text{coord.}}\ \vec{b}$$

Lin trans:

$$P_B : \mathbb{R} \longrightarrow V \qquad B = \{\vec{b}\}.$$
$$a \mapsto a\vec{b}$$

① one-to-one

② onto

$\Big\} \longrightarrow$ "P_B^{-1}" exists.

$$P_B^{-1} : V \to \mathbb{R}$$

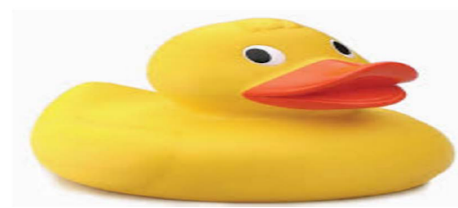
$$\vec{v} \mapsto \underbrace{[\vec{v}]_B}_{} \in \mathbb{R}$$

Coordinate of $\vec{v}$ w.r.t. B.

$\equiv$ B-coordinate of $\vec{v}$

Ex. V = collection of all possible ducks differing only by height & width

. . . .



Check.

$\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$ is a basis for V

$\vec{b_1}$

$\vec{b_2}$

Coordinate: KEY.

$$V \sim \mathbb{R}^n.$$

$\dim V = n < \infty$

Ex. $V = \mathbb{P}_2$. Find coordinates of

$f(x) = x^2 + 3x + 2$ w.r.t.

(1) $B_1 = \{1, x, x^2\}$

(2) $B_2 = \{x, 1, 1-x^2\}$.

(1) $[f]_{B_1} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

(2) $f(x) = a_1 x + a_2 \cdot 1 + a_3 (1-x^2)$

$\xrightarrow{\text{lin. sys.}} \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow{\text{sol.}} [f]_{B_2}$.

$\underline{Ex}$. $\vec{v} \in V = \mathbb{R}^2$. Basis. B, C.

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad [\vec{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$C = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}. \quad [\vec{v}]_C = ?$$

$$\vec{v} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\|$$

$$\vec{v} = a_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Solve $\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 2 & 0 \end{array}\right] \to [\vec{v}]_C = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$\dim V = n$

$$\begin{tikzcd} & V \arrow[dr, "P_C^{-1}"] & \\ \underset{B}{\mathbb{R}^n} \arrow[ur, "P_B"] \arrow[rr, "P_{C\leftarrow B} = P_C^{-1} P_B"] & & \underset{C}{\mathbb{R}^n} \end{tikzcd}$$

$\downarrow$

change of coordinate.