Lec 12

$$[\vec{v}]_C = \underset{C \leftarrow B}{P} [\vec{v}]_B$$

$$= P_C^{-1} \underbrace{\left(P_B \left([\vec{v}]_B \right) \right)}_{\vec{v}}$$

Change of coordinate

$$\begin{tikzcd} & V \arrow[dl, shift left, "P_B^{-1}"] \arrow[dr, shift left, "P_C^{-1}"] & \\ \mathbb{R} \arrow[ur, shift left, "P_B"] & & \mathbb{R} \arrow[ul, shift left, "P_C"] \end{tikzcd}$$

 $= \vec{v}$

$\mathcal{B} = \{$  $\}$

$C = \{$  $\}$

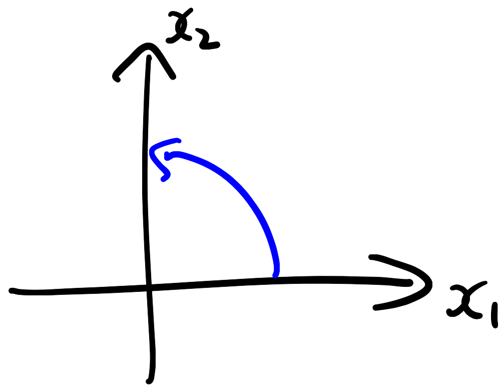
$[\vec{v}]_{\mathcal{B}} = 2$

$[\vec{v}]_C = -\frac{1}{2}$

$$[\vec{v}]_C = -\frac{1}{4}\,[\vec{v}]_B$$

$\underset{C \leftarrow B}{P}$

2D Rotation by 90° counterclockwise T



$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Standard matrix. $\quad A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T(\vec{x}) = A\vec{x}$$

What is standard matrix for $\underset{C \leftarrow B}{P}$?

$$\underset{C \leftarrow B}{P}(\vec{e}_1) = P_C^{-1}\left(P_B(\vec{e}_1)\right)$$

$$= P_C^{-1}(\vec{b}_1)$$

$$\equiv [\vec{b}_1]_C \longrightarrow \text{computation.}$$

$$P_B(\vec{e}_1) = [\vec{b}_1 \ \cdots \ \vec{b}_n] \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{b}_1$$

Standard matrix

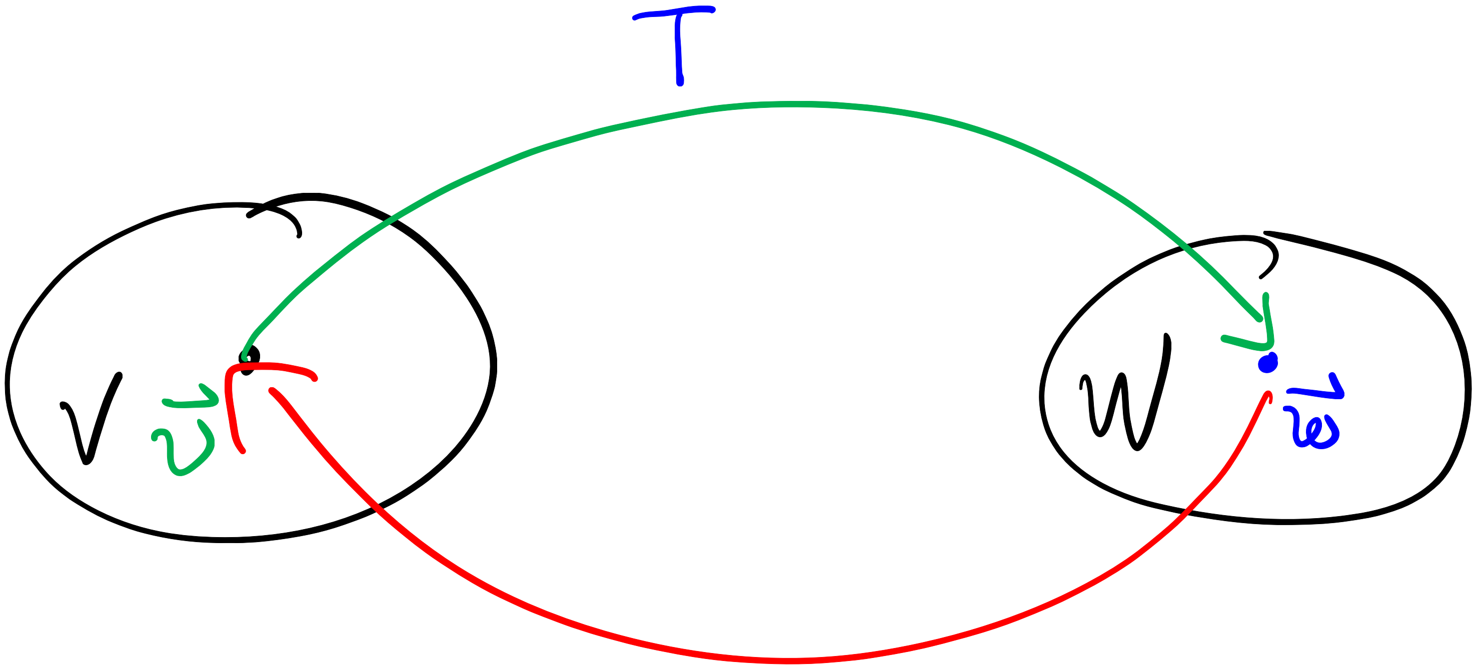
$$\underset{C \leftarrow B}{P} = \begin{bmatrix} [\vec{b_1}]_C & [\vec{b_2}]_C & \cdots & [\vec{b_n}]_C \end{bmatrix}$$

Generalization of matrix inverse.

Def. Lin. trans. $V \to W$ is a (linear) isomorphism if there is another lin. trans. $T^{-1}: W \to V$ s.t.

$$T(T^{-1}(\vec{w})) = \vec{w}, \quad \forall \vec{w} \in W$$

$$T^{-1}(T(\vec{v})) = \vec{v}, \quad \forall \vec{v} \in V$$



iso morphism

↓ ↓

equal shape.

Coordinate.

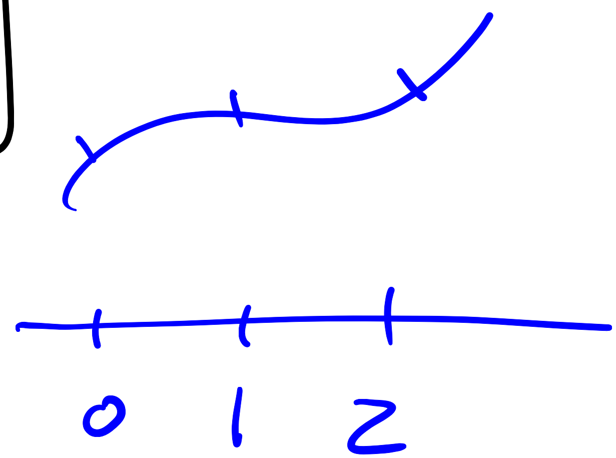
$$P_B : \mathbb{R}^n \longrightarrow V$$

inverse P_B^{-1}

P_B is an isomorphism.

Ex. Is $T: \mathbb{P}_2 \to \mathbb{R}^3$ an isomorphism?

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$



① $T(a\,p(x) + b\,q(x))$

$$= \begin{bmatrix} a\,p(0) + b\,q(0) \\ a\,p(1) + b\,q(1) \\ a\,p(2) + b\,q(2) \end{bmatrix} = a \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix} + b \begin{bmatrix} q(0) \\ q(1) \\ q(2) \end{bmatrix}$$

$= aT(p(x)) + b\,T(q(x))$

T is lin. trans.

② $p(x) = a_0 + a_1 x + a_2 x^2$

$$T(p(x)) = \begin{bmatrix} a_0 \\ a_0 + a_1 + a_2 \\ a_0 + 2a_1 + 4a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{\text{invertible.}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$\Rightarrow$ Yes. Isomorphism.

Ex. $T: \mathbb{P}_2 \to \mathbb{R}^3$.

$$T(p(x)) = \begin{bmatrix} p'(0) \\ p'(1) \\ p'(2) \end{bmatrix}$$

Is T isomorphism?

$$T(c) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

NOT one to one

$\Rightarrow$ NOT isomorphism.

Matrix representation of general lin. trans.

$$T : V \longrightarrow W$$

V : $\quad \mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ $\quad \dim V = n$

W : $\quad C = \{\vec{c}_1, \cdots, \vec{c}_m\}$. $\quad \dim W = m$

$$
T: \begin{tikzcd}
V \arrow[r] & W \arrow[d, "P_C^{-1}"] \\
\mathbb{R}^n \arrow[u, "P_B"] \arrow[r, "A = P_C^{-1} T P_B"] & \mathbb{R}^m \\
{[\vec{v}]_B} & {[T(\vec{v})]_C}
\end{tikzcd}
$$

matrix representation.

Find standard matrix.

$$(P_C^{-1} T P_B)(\vec{e}_1) = P_C^{-1}\left(T(\vec{b}_1)\right)$$

$$P_B(\vec{e}_1) = \vec{b}_1 \qquad = [T(\vec{b}_1)]_C$$

$$A = \left[\,[T(\vec{b}_1)]_C \;\cdots\; [T(\vec{b}_n)]_C\,\right] \in \mathbb{R}^{m \times n}.$$