

Lec 14. determinant .

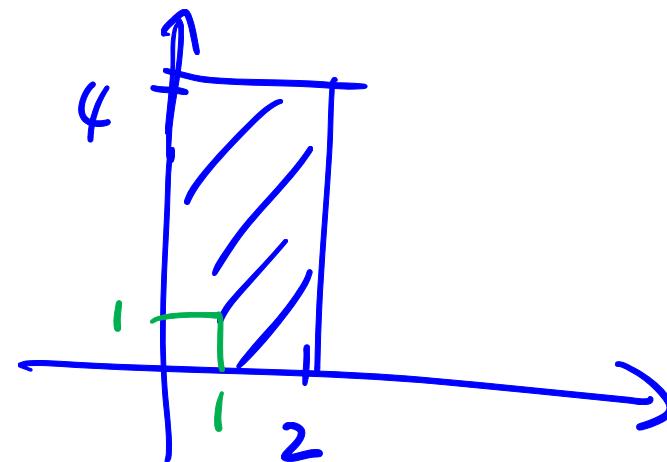
- 1) Whether a matrix is invertible.
 - 2) eigenvalues. (chap 5).
- $\det(A - \lambda I)$
- 3) Quantum mechanics (for fermions)
 - 4) Statistics & machine learning .

(e.g. determinantal point process)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

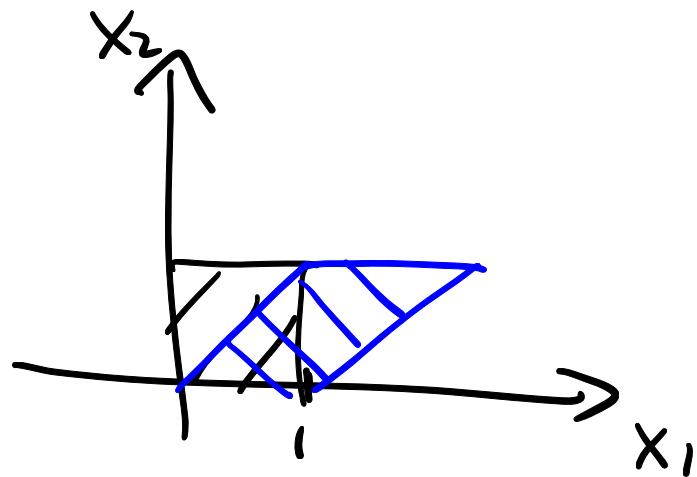
$$\det(A) = ad - bc$$

ex. $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$



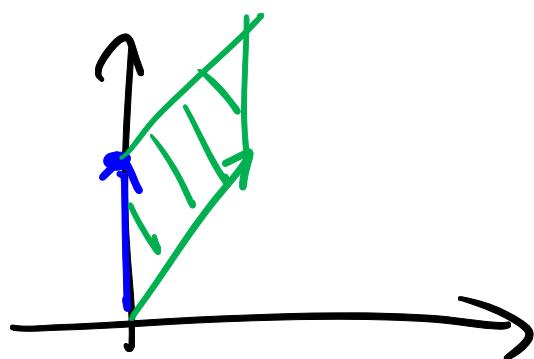
$$\det(A) = 8.$$

Ex. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



$$\det(A) = 1.$$

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$



$$\det(A) = -1$$

$\det(A)$ v.s. elementary row op.

$$\det(A) = ad - bc. \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- ① add a scalar multiple of a row
to another row

$$\det \begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix} \rightarrow \text{generalization of "shear"}$$

$$= (a+kc)d - (b+kd)c = \det(A)$$

② exchange 2 rows.

$$\det \begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -\det(A).$$

③ Scale any row by k ($k \neq 0$).

$$\begin{aligned} \det \begin{pmatrix} ak & bk \\ c & d \end{pmatrix} &= k(ad - bc) \\ &= k \cdot \det(A). \end{aligned}$$

$A \xrightarrow{\text{elem. row op.}} A' (\text{RREF})$

RREF	$\det(A')$
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0

$$\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right] \quad | \quad 1 \leftarrow \text{invertible} .$$

$\det(A) = \text{Some nonzero number}$

$$\times \det(A')$$

$$\det(A) \neq 0 \Leftrightarrow \det(A') \neq 0$$

$\Leftrightarrow A \text{ is invertible} .$

Ex. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

$$\det(A) = 3$$

$$\det(B) = 2$$

$$AB = \begin{bmatrix} 3 \cdot 2 + 0 \cdot 2 & 3 \cdot 1 + 0 \cdot 2 \\ 6 \cdot 2 + 1 \cdot 2 & 6 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 14 & 8 \end{bmatrix}$$

$$\det(AB) = 6 \cdot 8 - 3 \cdot 14 = 6 = \det(A) \cdot \det(B)$$

Fact: $A, B \in \mathbb{R}^{n \times n}$

$$\det(AB) = \det(A) \cdot \det(B)$$

As an application, A is invertible.

B is not invertible.

$\Rightarrow AB$ is **NOT** invertible

$\det(A) \neq 0$. $\det(B) = 0$.

$\Rightarrow \det(AB) = 0 \Rightarrow$ NOT invertible .

det of 3×3 matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

The diagram shows a 3×3 matrix with columns labeled a_1, a_2, a_3 and rows labeled b_1, b_2, b_3 . The entries are a_1, a_2, a_3 in the first row; b_1, b_2, b_3 in the second row; and c_1, c_2, c_3 in the third row. Blue lines connect the entries a_1, a_2, a_3 to the entries b_1, b_2, b_3 respectively. Red lines connect the entries b_1, b_2, b_3 to the entries c_1, c_2, c_3 respectively. Dashed red lines extend from the bottom-right corner c_3 towards the top-left corner a_1 , forming a diagonal line across the matrix.

$$\det(A) = +a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

$$- a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Ex. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$$\det(A) = 1 \cdot 3 \cdot (-2) + 2 \cdot 1 \cdot (-1) + 0$$

$$- 1 \cdot 3 \cdot (-1) - 0 - 0$$

$$= -6 - 2 + 3 = -5.$$

General $n \times n$ matrix

In due time definition.

Start from $n=1$.

$$A = [a_{11}] \quad \det(A) = a_{11}$$

assume we know how to
define $\det(A)$ for any matrix
of size $(n-1)$

$\det(A) = ?$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ ; & & ; & & ; \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ ; & & ; & & ; \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

ith row *Col j*

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & \cancel{a_{ij}} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \cancel{a_{i1}} & \cdots & \cancel{a_{ii}} & \cdots & \cancel{a_{in}} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & \cancel{a_{nj}} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{(n-1) \times (n-1)}$$

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) \cdots + (-1)^{n+1} a_{1n} \det(A_{1n})$$

Ex. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = a \cdot d - b \cdot c$$

