Lec 14. determinant.

1) Whether a matrix is invertible.

2) eigenvalues. (chap 5).

$$\det(A-\lambda I)$$

3) Quantum mechanics (for fermions)

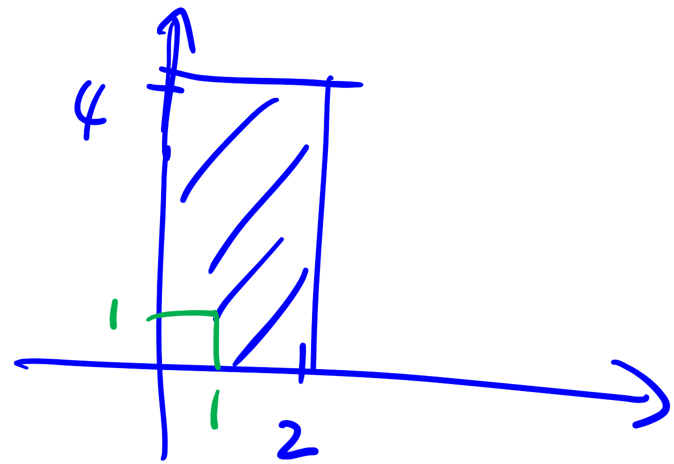
4) Statistics & machine learning.

(e.g. determinantal point process)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

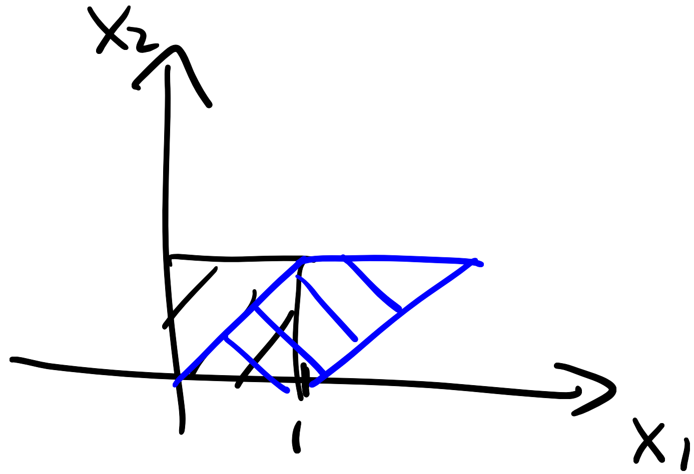
$$\det(A) = ad - bc$$

<u>ex</u>. $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$



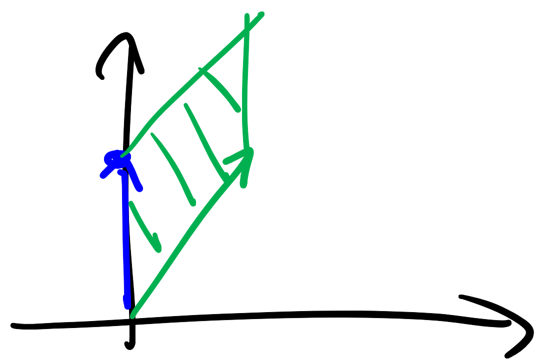
$$\det(A) = 8.$$

Ex. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



$\det(A) = 1.$

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$



$\det(A) = -1$

det(A) v.s. elementary row op.

$\det(A) = ad - bc$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

① add a scalar multiple of a row to another row

$$\det\begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}$$

→ generalization of "shear"

$$= (a+kc)d - (b+kd)c = \det(A)$$

② exchange 2 rows.

$$\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -\det(A).$$

③ Scale any row by k $(k \neq 0)$.

$$\det\begin{pmatrix} ak & bk \\ c & d \end{pmatrix} = k(ad-bc)$$
$$= k \cdot \det(A).$$

$$A \xrightarrow{\text{elem. row op.}} A' \text{ (RREF)}$$

RREF	$\det(A')$
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0

$\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\quad 1 \leftarrow$ invertible.

$\det(A) =$ Some nonzero number

$\times \det(A')$

$\det(A) \neq 0 \iff \det(A') \neq 0$

$\iff$ A is invertible.

Ex. $A = \begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

$\det(A) = 3$

$\det(B) = 2$

$$AB = \begin{bmatrix} 3\cdot 2 + 0\cdot 2 & 3\cdot 1 + 0\cdot 2 \\ 6\cdot 2 + 1\cdot 2 & 6\cdot 1 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 14 & 8 \end{bmatrix}$$

$$\det(AB) = 6\cdot 8 - 3\cdot 14 = 6 = \det(A)\cdot\det(B)$$

Fact: $A, B \in \mathbb{R}^{n\times n}$

$$\det(AB) = \det(A) \cdot \det(B).$$

As an application, A is invertible.

B is not invertible.

$\Rightarrow AB$ is NOT invertible

$\det(A) \neq 0$. $\det(B) = 0$.

$\Rightarrow \det(AB) = 0 \Rightarrow$ NOT invertible.

<u>det of 3x3 matrix</u>

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$\det(A) = +a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2$$

$$- a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

$\underline{Ex}$. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$$\det(A) = 1 \cdot 3 \cdot (-2) + 2 \cdot 1 \cdot (-1) + 0$$

$$- 1 \cdot 3 \cdot (-1) - 0 - 0$$

$$= -6 - 2 + 3 = -5.$$

General $n \times n$ matrix

Inductive definition.

Start from $n=1$.

$A = [a_{11}]$ $\quad \det(A) = a_{11}$

assume we know how to define det(A) for any matrix of size $(n-1)$

$\det(A) = ?$

$$A = \begin{array}{r c}
 & \begin{array}{ccccc} \phantom{a_{11}} & & \text{col } j & & \phantom{a_{1n}} \end{array} \\
\begin{array}{r} \\ \\ i\text{th row} \\ \\ \\ \end{array} &
\begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & & \vdots & & \vdots \\
a_{n1} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}
\end{array} \in \mathbb{R}^{n\times n}$$

$$A_{ij} = \begin{bmatrix} a_{11} & \cdots & \cancel{a_{1j}} & \cdots & a_{1n} \\ \vdots & & \cancel{\vdots} & & \vdots \\ \cancel{a_{i1}} & \cancel{\cdots} & \cancel{a_{ij}} & \cancel{\cdots} & \cancel{a_{in}} \\ \vdots & & \cancel{\vdots} & & \vdots \\ a_{n1} & \cdots & \cancel{a_{nj}} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$$

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12})$$

$$+ a_{13}\det(A_{13}) \cdots + (-1)^{n+1} a_{1n}\det(A_{1n})$$

Ex. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = a \cdot d - bc$$