Lec 15. Cofactor expansion

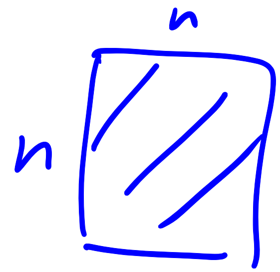
Cramer's rule.

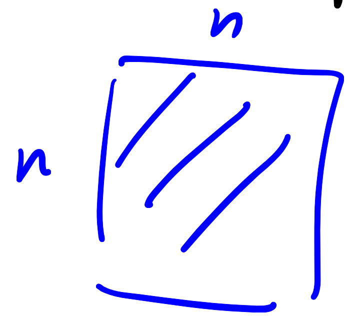
<u>Ex</u>. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$$\det(A) \equiv |A| = 1 \cdot \begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 3 \\ -1 & 0 \end{vmatrix}$$

$$= 1 \cdot (-6) - 2 \cdot 1 + 1 \cdot 3 = -5.$$

Cofactor. $A \in \mathbb{R}^{n\times n}$. $C \in \mathbb{R}^{n\times n}$





$$\underbrace{[C]_{ij}}_{\substack{\text{(i,j)th entry}\\ \text{of } C}} = (-1)^{i+j}\,|\underbrace{A_{ij}}_{\text{minor. } \mathbb{R}^{(n-1)\times(n-1)}}|$$

Repeat def.

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

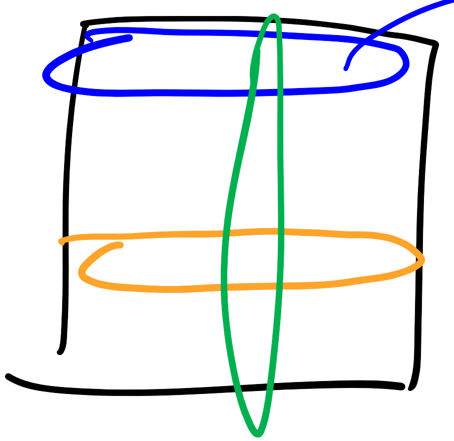
<u>Fact</u> (Cofactor expansion).

(1) expand over any row. $\forall\, 1 \le i \le n$.

$$\det(A) = a_{i1} C_{i1} + \cdots + a_{in} C_{in}$$

(2) expand over any column $\forall\, 1 \le j \le n$.

$$\det(A) = a_{1j} C_{1j} + \cdots + a_{nj} C_{nj}.$$



what we learned earlier

VERY USEFUL if some row/col has a lot of 0's.

Ex. $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$

$$|A| = (-1)\cdot(-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + (-2)(-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$$

$$= +1 + (-2)\cdot 3 = -5.$$

Ex.

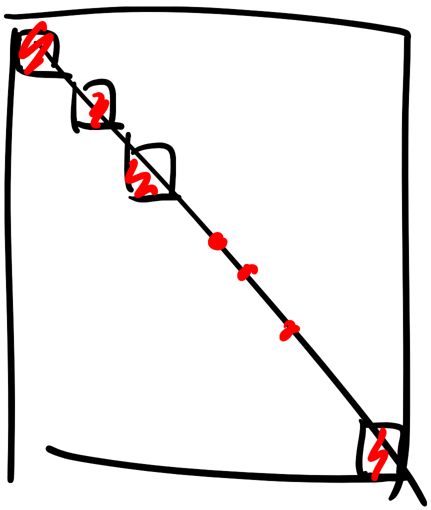
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ & & \vdots & \\ 0 & 0 & \cdots \; 0 & a_{nn} \end{vmatrix}$$

upper triangular.

$$= a_{11} \cdot (-1)^{1+1} \begin{vmatrix} a_{22} & \cdots & & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ & & \ddots & \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

$$= a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}.$$

When is an upper triangular matrix invertible?



$a_{ii} \neq 0 \quad , \forall\ 1 \leq i \leq n.$

$$\det(A) = \prod_{i=1}^{n} a_{ii} \neq 0 .$$

product

Ex.

$$\begin{vmatrix} 3 & -7 & 8 & 0 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 2 & 0 \end{vmatrix}$$

upper triangular!

$$= 2 \cdot (-1)^{5+4} \cdot \begin{vmatrix} 3 & -7 & 8 & -6 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -2 \cdot 3 \cdot 2 \cdot 1 \cdot (-1) = 12$$

Ex. $\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix}$ lower triangular

$$= a_{11} \cdot (-1)^{1+1} \begin{vmatrix} a_{22} & & 0 \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \prod_{i=1}^{n} a_{ii}$$

FACT. $\det(A) = \det(A^T)$. $\forall A \in \mathbb{R}^{n\times n}$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & \cdots & \vdots \\ & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$$

$$|A| = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$$

cofactor over 1st row

$$|A^T| = a_{11}\tilde{C}_{11} + a_{12}\tilde{C}_{21} + \cdots + a_{1n}\tilde{C}_{n1}.$$

$$\rightarrow \quad C_{1j} = \tilde{C}_{j1} \Rightarrow |A| = |A^T|.$$

determinant & elementary row op.

Ex. exchange of 2 rows.

WLOG. Consider first 2 rows.

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$\parallel$ cofactor 1st row.

$\parallel$ cofactor 2nd row

$$a_{11}C_{11} + \cdots + a_{1n}C_{1n} \qquad (-1)(a_{11}C_{11} + \cdots + a_{1n}C_{1n})$$

Ex. $|A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{11} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$

$$|A| = -|A| \Rightarrow |A| = 0,$$

Cramer's rule: Solve $A\vec{x} = \vec{b}$ using determinants.

$$I_i = \begin{array}{c} \begin{bmatrix} 1 & 0 & x_1 & 0 & 0 \\ 0 & 1 & \vdots & \vdots & \vdots \\ \vdots & \vdots & x_i & & \vdots \\ & & & 1 & \vdots \\ 0 & 0 & x_n & 0 & 0 \\ & & & 0 & 1 \end{bmatrix} \\ \begin{array}{ccccc} \vec{e}_1 & \vec{e}_2 & \uparrow \text{ ith col.} & \vec{e}_{i+1} \cdots & \vec{e}_n \end{array} \end{array} \begin{array}{l} \\ \\ \leftarrow \text{ ith row} \\ \\ \\ \\ \\ \end{array}$$

$$|I_i| = x_i \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = x_i$$

$$A\, I_i = [A\vec{e}_1 \;\; A\vec{e}_2 \cdots A\vec{x} \cdots A\vec{e}_n]$$

$$= [\vec{a}_1 \;\; \vec{a}_2 \;\; \vec{b} \;\; \vec{a}_n]$$

Take det.

$$|A\, I_i| = |A| \cdot \underbrace{|I_i|}_{= x_i} = |\vec{a}_1 \cdots \underbrace{\vec{b}}_{i\text{th col}} \cdots \vec{a}_n|$$

$$\Rightarrow \quad x_i = \frac{1}{\det(A)} \cdot \det(\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n) \qquad \forall i$$