Lec 16. A formula for matrix inverse.

eigenvalue.

Ex. (Cramer's rule).

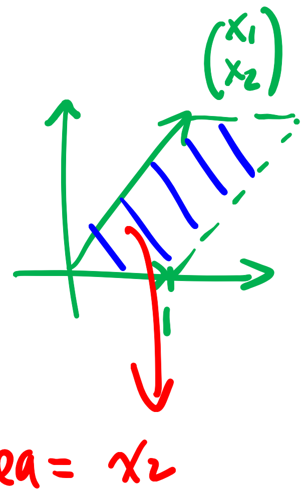
$A\vec{x} = \vec{b}$. $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 4 \end{array}\right]$ $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}} = \frac{2}{2} = 1$$

$$x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix}}{2} = \frac{4}{2} = 2.$$

Geometric interpretation.

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & x_1 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{b} \end{bmatrix}.$$



$$x_2 = \frac{\det(\vec{a}_1 \;\; \vec{b})}{\det(\vec{a}_1 \;\; \vec{a}_2)}$$

A formula for A^{-1}

$$A\vec{x}_i = \vec{e}_i \ , \ i = 1, \cdots, n$$

$$A^{-1} = [\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n].$$

Apply Cramer's rule to find all entries of $\underbrace{[A^{-1}]_{ij}}$,

NOT minor,

just (i,j)-th entry of A^{-1}.

$[A^{-1}]_{11}$: $A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$(\vec{x}_1)_1 = \frac{1}{\det(A)} \cdot \begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & a_{n2} & & a_{nn} \end{vmatrix}$$

$$= \frac{1}{\det A} \cdot C_{11} \leftarrow \text{cofactor.}$$

$$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n2} & & a_{nn} \end{vmatrix}$$

$[A^{-1}]_{12}$: $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$(\vec{X_2})_1 = \frac{1}{|A|}\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{|A|}(-1)^{1+2}\begin{vmatrix} a_{12} & \cdots & a_{1n} \\ a_{32} & \cdots & a_{3n} \\ \vdots & \ddots & \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \frac{1}{|A|} C_{21}$$

$$[A^{-1}]_{21} : \quad A\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$(\vec{x_1})_2 = \frac{1}{|A|}\begin{vmatrix} a_{11} & 1 & \cdots & a_{1n} \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{n1} & 0 & \cdots & a_{nn} \end{vmatrix} = \frac{1}{|A|} C_{12}$$

In general.

$$A^{-1} = \frac{1}{|A|} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ \vdots & & \ddots & \vdots \\ C_{1n} & & \cdots & C_{nn} \end{bmatrix}}_{\downarrow}$$

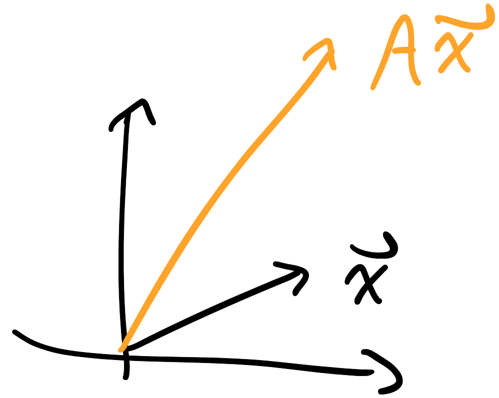
C^T. $\leftarrow$ adjugate of A

$\mathrm{adj}(A)$.

Eigenvalue & eigenvectors.

$A \in \mathbb{R}^{n \times n}$.

random $\vec{x} \in \mathbb{R}^n$



Change in both length and direction.

eigenvector: $\vec{x}$ s.t. $A\vec{x}$ does not change direction.

$$A\vec{x} = \lambda \vec{x}$$

$\vec{x} \longrightarrow$ eigenvector

λ: eigenvalue.

<u>Ex</u>. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\vec{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{v}_1$

eigenvector

eigenvalue.

$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $A\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\vec{v}_2$

$[\vec{v}_1, \vec{v}_2]$ lin. indep. basis of $\mathbb{R}_2$.

Ex. $A^{1000}\begin{bmatrix}1\\1\end{bmatrix} = A^{999}\cdot\left(A\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$= A^{999}\cdot 3\begin{bmatrix}1\\1\end{bmatrix}$$

$$= A^{998}\cdot 3^2\begin{bmatrix}1\\1\end{bmatrix} = \cdots$$

$$= 3^{1000}\begin{bmatrix}1\\1\end{bmatrix}.$$

$$A^{1000}\cdot\begin{bmatrix}1\\-1\end{bmatrix} = (-1)^{1000}\cdot\begin{bmatrix}1\\-1\end{bmatrix}$$

$$A^{1000}\vec{x}$$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

$$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}(\vec{x}) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \leftarrow \text{coordinate w.r.t. } \mathcal{B}.$$

$$\vec{x} = z_1 \vec{v_1} + z_2 \vec{v_2}$$

$$A^{1000}\vec{x} = A^{999} \cdot \left(z_1 A\vec{v_1} + z_2 A\vec{v_2} \right)$$

$$= A^{999} \cdot \left(z_1 \cdot 3\, \vec{v_1} + z_2 \cdot (-1)\, \vec{v_2} \right)$$

$$= \cdots$$

$$= z_1 \cdot 3^{1000} \vec{v}_1 + z_2 \cdot (-1)^{1000} \vec{v}_2$$