Lec 17.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$AV = \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$AV = \underset{\sim}{V} D$$

$\rightarrow$ invertible.

$\Rightarrow \quad D = V^{-1} A V$

diagonalization.

How to systematically find eigenvalues & eigenvectors?

$$A\vec{v} = \lambda \vec{v}$$

$$\Rightarrow (A - \lambda I)\vec{v} = \vec{0} \quad . \quad \vec{v} \neq \vec{0}$$

homogeneous lin. eq.

has a nontrivial sol.

$\Rightarrow A - \lambda I$ is NOT invertible

$$\Rightarrow \boxed{\det(A - \lambda I) = 0}$$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0.$$

$$\parallel$$

$$(1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = -1$.

$\lambda_1 = 3.$ $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -1.$ $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

What could happen during diagonalization?

Q: Are all eigenvalues different?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0.$$

2 roots. $\lambda_1 = \lambda_2 = 1$.

$\lambda = 1$ has multiplicity 2.

$(A - \lambda I)\vec{v} = \vec{0}$.

$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Pick any 2 lin. indep. vectors $\vec{v_1}, \vec{v_2} \in \mathbb{R}^2$.

to diagonalize A. $(V^{-1} \cdot I \cdot V = I)$.

$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1+\epsilon \end{bmatrix}$ $\epsilon \neq 0$.

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & (1+\epsilon)-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(1+\epsilon-\lambda) = 0$

$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 1+\epsilon.$

$\epsilon \to 0. \quad \lambda_1 = \lambda_2 = 1.$ multiplicity 2.

This perspective is important for understanding 2nd order ODEs.

Q: Are all matrices diagonalizable?

Ex. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0.$$

$\lambda_1 = \lambda_2 = 2$. has multiplicity 2.

To find eigenvectors.

$$\begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} \vec{v} = \vec{0}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

sol set is one dimensional.

$\Rightarrow$ A is NOT diagonalizable!

Perturbative view point.

$$\tilde{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2+\epsilon \end{bmatrix}.$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2+\epsilon-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow \quad \lambda_1 = 2, \quad \lambda_2 = 2 + \epsilon.$$

$$\lambda_1 = 2: \quad \begin{bmatrix} 0 & 1 \\ 0 & \epsilon \end{bmatrix} \vec{v}_1 = \vec{0}. \qquad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2 + \epsilon. \quad \begin{bmatrix} -\epsilon & 1 \\ 0 & 0 \end{bmatrix} \vec{v}_2 = \vec{0}. \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$$

$$\epsilon \to 0. \qquad \vec{v}_1 = \vec{v}_2 \; !$$

Ex. $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & & a_{nn} \end{bmatrix}$ upper triangular

Find eigenvalues of A.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & & a_{nn}-\lambda \end{vmatrix} = 0.$$

$$\|$$

$$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

$\lambda_i = a_{ii}, \quad i = 1, \cdots, n$. (Counting multiplicity)

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

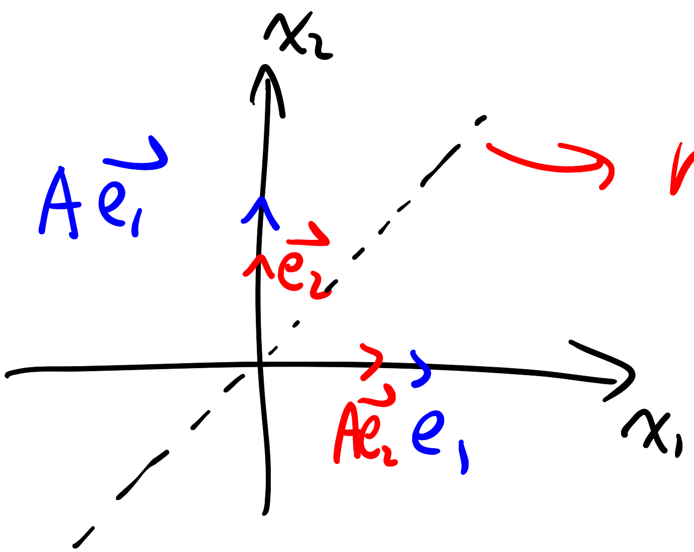
$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 - 1 = 0$.

$\lambda_1 = 1, \quad \lambda_2 = -1$.

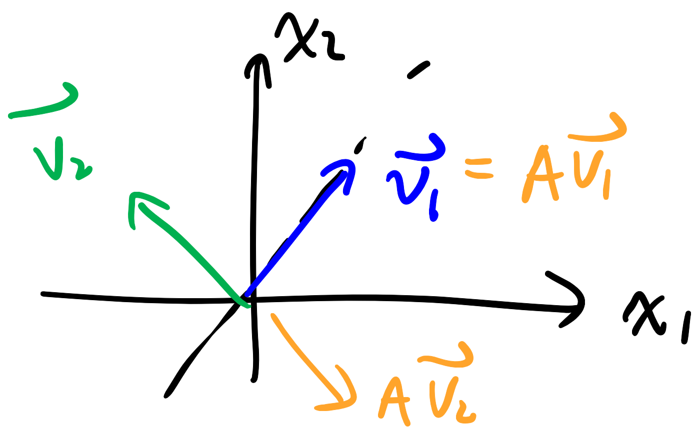
For $\lambda_1 = 1$. $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1$, $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

geometric meaning.



reflect w.r.t. $x_1 = x_2$



? eigenvalues of a rotation matrix ?