Lec 18. Complex eigenvalues. diagonalizability.

Warm up: Find eigenvalues / eigenvectors of

(1) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ (2) $A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

(1) $\begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0. \Rightarrow \lambda_1 = 1, \lambda_2 = 2$

For $\lambda_1 = 1$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda_2 = 2$, $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$V = [\vec{v}_1, \vec{v}_2]$. V is invertible

$AV = VD$. $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$$\underline{D = V^{-1} A V}.$$

(2) $\begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0$.

$\Rightarrow \lambda_1 = \lambda_2 = 2$.

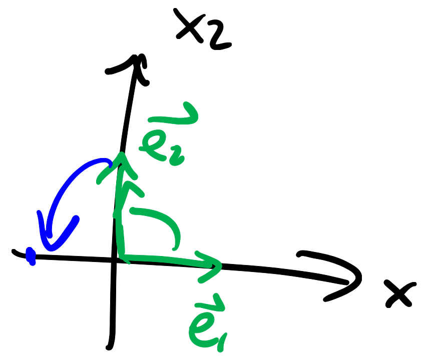
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \vec{v} = \vec{0}$ $\Rightarrow$ The sol set is $\left\{ \begin{bmatrix} 0 \\ c \end{bmatrix} \middle| c \in \mathbb{R} \right\}$.

we can pick $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as an eigenvector.

$\dim \mathrm{Null}(A - 2I) = 1$.

A is NOT diagonalizable.

Ex. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.



counter clockwise rotation of $\frac{\pi}{2}$.

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0. \Rightarrow \lambda = \pm i$$

$\downarrow$

imaginary unit.

For $\lambda_1 = i$. $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \vec{v} = \vec{0}$

$$\begin{cases} -i\,v_1 - v_2 = 0 \\ v_1 - i\,v_2 = 0 \end{cases} \longrightarrow \begin{cases} v_1 - i\,v_2 = 0 \\ \cancel{v_1 - i\,v_2 = 0} \end{cases}.$$

sol set is $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} c \;\middle|\; c \in \mathbb{C} \right\}$

we can pick $\vec{v_1} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (in other words,

we can also choose $c = -2i$, $\vec{v_1} = \begin{bmatrix} 2 \\ -2i \end{bmatrix}$)

For $\lambda_2 = -i$, $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \vec{v_2} = \vec{0}$.

sol set is $\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} c \;\middle|\; c \in \mathbb{C} \right\}$.

we can pick $\vec{v_2} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

(we can also choose $c = i$, $\vec{v_2} = \begin{bmatrix} i \\ -1 \end{bmatrix}$)

Ex. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in \mathbb{R}.$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0.$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0.$$

$$\Rightarrow \lambda^2 - 2\cos\theta\lambda + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 0$$

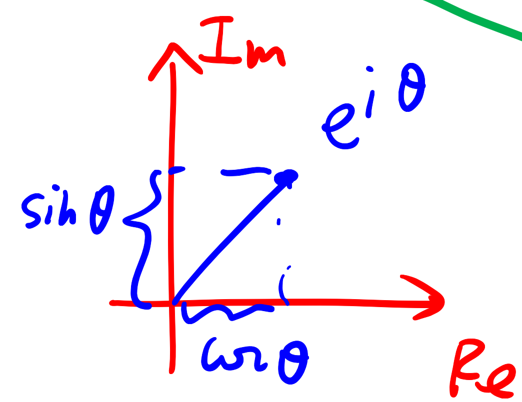
$$\Rightarrow \lambda^2 - 2\cos\theta\ \lambda + 1 = 0.$$

$$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - (\cos^2\theta + \sin^2\theta)}$$

$$= \cos\theta \pm i\sin\theta.$$

Euler's formula.



$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\lambda_1 = e^{i\theta}, \quad \lambda_2 = e^{-i\theta}.$$

Diagonalizability.

Def $A \in \mathbb{R}^{n\times n}$. If we can find n lin. indep. eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$ s.t.

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad i=1,\cdots,n.$$

Then A is diagonalizable.

$$A\underbrace{\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}}_{V} = \begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}\underbrace{\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}}_{D}$$

$$\Rightarrow AV = VD \quad \text{or} \quad D = V^{-1}AV.$$

Fact. A is diagonalizable.

$\Rightarrow A^T$ is ".

Proof: $A = VDV^{-1}$.

Apply transpose on both sides.

$$A^T = (VDV^{-1})^T = (V^{-1})^T D V^T$$

$$= (V^T)^{-1} D (V^T)$$

V invertible $\Rightarrow (V^T)$ is also invertible $\square$

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eigenvalues of A and A^T are the same.

another way of viewing this.

$$0 = |A^T - \lambda I| = |(A^T - \lambda I)^T| = |A - \lambda I|.$$

(this is true even if A is NOT diagonalizable)