Lec 19.

T/F: There exists $A \in \mathbb{R}^{n\times n}$ with $(n+1)$ eigenvalues (counting multiplicity).

False.

$$0 = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ & & \ddots & \vdots \\ \cdots & \cdots & & a_{nn}-\lambda \end{vmatrix} = \underbrace{(a_{11}-\lambda)\cdot \overbrace{\begin{vmatrix} a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \ddots & \\ a_{n2} & & a_{nn}-\lambda\end{vmatrix}}^{\in \mathbb{P}_{n-1}}}_{\in \mathbb{P}_n}$$

$$\left.\begin{aligned} & - a_{12}\cdot \begin{vmatrix} * \end{vmatrix} \\ & + \cdots + (-1)^{n+1} a_{1n} \cdot \begin{vmatrix} * \end{vmatrix} \end{aligned}\right\} \in \mathbb{P}_{n-1}$$

Characteristic polynomial of deg n.

$0 = c_0 + c_1\lambda + \cdots + c_n\lambda^n$ has exactly n roots in $\mathbb{C}$ (counting multiplicity)

Diagonalizability.

$A \in \mathbb{R}^{2\times 2}$. $A\vec{v_1} = \lambda_1 \vec{v_1}$ $\lambda_1 \neq \lambda_2$.

$A\vec{v_2} = \lambda_2 \vec{v_2}$

$\hookrightarrow$ distinct eigenvalue.

Claim $\{\vec{v_1}, \vec{v_2}\}$. lin. indep.

Pf: Assume the statement is not true.

then $c_1\vec{v_1} + c_2\vec{v_2} = \vec{0}$ WLOG.

i.e. $\vec{v}_1 = c\,\vec{v}_2$ for some c.

$$\Rightarrow \underbrace{(A-\lambda_1 I)\vec{v}_1}_{=0} = \underbrace{(A-\lambda_1 I)\,c\,\vec{v}_2}_{=\underbrace{(\lambda_2-\lambda_1)}_{\neq 0}\cdot c\,\vec{v}_2}.$$

$\Rightarrow c=0.$ $\Rightarrow \vec{v}_1 = \vec{0}$.

This is a contradiction to the fact that $\vec{v}_1$ is an eigenvector. $\square$.

Fact $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues then A is diagonalizable.

Ex. $A = \begin{bmatrix} a_{11} & * & \cdots & * \\ & a_{22} & & \vdots \\ & & \ddots & * \\ 0 & & & a_{nn} \end{bmatrix}$

$a_{11}, \cdots, a_{nn}$ are n distinct numbers.

$\Rightarrow$ they are n distinct eigenvalues of A

$\Rightarrow A$ is diagonalizable.

T/F: All upper triangular matrices are diagonalizable.

False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

2x2: $\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$ for any a. NOT diag.

3x3:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ has 2 lin. ind. eigenvectors.

$\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ has 1 lin. ind. eig.vec.

$\lambda = a$ (w. multiplicity 3) is the eigenvalues.

$$\left[\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

free

sol set is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot c \;\middle|\; c \in \mathbb{C} \right\}.$$

4 x 4 .

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$$ 3 eig. vec.

$$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$$ 2 eig. vec.

$$\begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$$ 1 eig. vec.

Fact. there exists arbitrarily large non-diagonalizable matrices.

Fact $A \in \mathbb{R}^{n \times n}$. λ is an eigenvalue.

$$1 \leq \underbrace{\dim \mathrm{Null}(A - \lambda I)}_{\text{geometric multiplicity of } \lambda} \leq \underbrace{\text{multiplicity of } \lambda}_{\text{algebraic multiplicity of } \lambda}.$$

$$\boxed{1 \le \text{Geo. multi} \le \text{Alg. multi}}$$

<u>Def</u>. $Null(A-\lambda I)$ is called the eigenspace of A w. eigenvalue λ.

eigenspace = span of all eigenvectors.

Ex. $A \in \mathbb{R}^{6\times 6}$.

$$|A-\lambda I| = (1-\lambda)^3(2-\lambda)\lambda^2.$$

possible total dimension of eigenspace.

$\lambda =$ 0, 1, 2.

algebraic multiplicity 2 3 1

$$1 \le \dim \operatorname{Null}(A) \le 2.$$

$$1 \le \dim \operatorname{Null}(A - I) \le 3.$$

$$\dim \operatorname{Null}(A - 2I) = 1.$$

Possible numbers are

3, 4. 5, 6