Lec 2. Solving lin. sys.

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = 3 \end{cases}$$

$x_3 = 3 \longrightarrow x_3 = 3$

$x_2 - 4x_3 = 4 \longrightarrow x_2 = 4\times 3 + 4 = 16$

$x_1 = 2\times 16 - 3 = 29$

sol set

$\{(29, 16, 3)\}$

augmented matrix of lin. sys.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

back substitution.

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right]$$

$\blacksquare$ nonzero. pivot

$*$ any value. $\Rightarrow$ sol. unique.

Ex. $\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ 2x_2 - 8x_3 = 8 & (2) \\ -4x_1 + 5x_2 + 9x_3 = -9. & (3) \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Row reduction algorithm.

$\frac{1}{2} \times (2)$:

$$x_2 - 4x_3 = 4 \quad (2')$$

$4 \times (1) + (3)$:

$$-3x_2 + 13x_3 = -9 \quad (3')$$

$3 \times (2') + (3')$:

$$x_3 = 3 \quad (3'')$$

$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ x_2 - 4x_3 = 4 & (2') \\ x_3 = 3 & (3'') \end{cases}$

(1) (2) (3)

$\Longleftrightarrow$ (1) (2') (3'')

Elementary row operations.

(R_1) Add a multiple of any row to any other row.

(R_2) Exchange 2 rows.

(R_3) Scale any row by nonzero number.

Ex. $\begin{cases} 2x_1 - 3x_2 \qquad\qquad - x_4 = 7 \\ x_1 + x_2 - 3x_3 + 2x_4 = 0. \end{cases}$

$$\left[\begin{array}{cccc|c} 2 & -3 & 0 & -1 & 7 \\ 1 & 1 & -3 & 2 & 0 \end{array}\right]$$

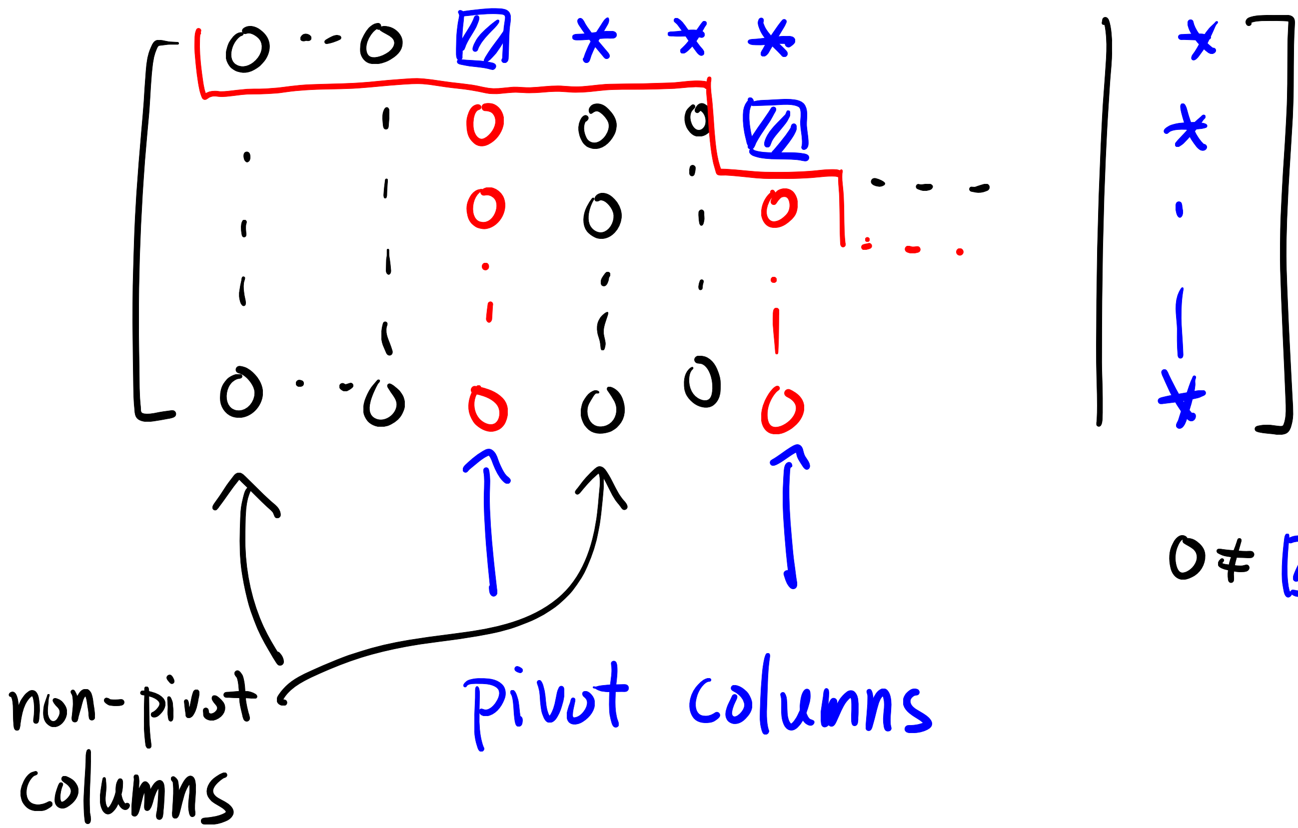
$\downarrow$ 2 × Row 2

$$\left[\begin{array}{cccc|c} 2 & -3 & 0 & -1 & 7 \\ 2 & 2 & -6 & 4 & 0 \end{array}\right]$$

$\downarrow$ add (-1) × Row 1 to Row 2.

$$\left[\begin{array}{cccc|c} \boxed{2} & -3 & 0 & -1 & 7 \\ 0 & \boxed{5} & -6 & 5 & -7 \end{array}\right]$$

Row echelon form (REF)



$0 \neq \boxed{}$ pivot positions

$$\left[\begin{array}{ccc|c} \boxed{} & * & * & * \\ 0 & \boxed{} & * & * \\ 0 & 0 & \boxed{} & * \end{array}\right]$$

Ex.

$$\left[\begin{array}{cccc|c} \boxed{2} & 0 & 1 & -1 & 5 \\ 0 & \boxed{3} & 0 & 0 & c \\ 0 & d & \boxed{-2} & 0 & 1 \end{array}\right]$$

For what values of c, d is this a REF?

$d = 0$. $c \in \mathbb{R}$.

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

$$\Updownarrow$$

$$\begin{cases} x_1 = * \\ x_2 = * \\ x_3 = * \end{cases}$$

back substitution:

convert REF

→ reduced REF (RREF)

$$\left[\begin{array}{cccccccc|c} 0 & \cdots & 0 & 1 & * & * & 0 & & * \\ \vdots & & \vdots & 0 & 0 & 0 & 1 & & * \\ & & & 0 & 0 & \vdots & 0 & \cdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & & * \end{array}\right]$$

pivot column

only has one nonzero entry $= 1$.

Thm Given any augmented matrix. we can find a RREF equivalent to the original augmented matrix by the row reduction alg.

Thm. RREF is unique.

Write lin. sys. interms of column vectors

<u>Def</u> An n-vector (a.k.a. vector of size n) is an ordered list of n numbers.

$$\vec{x} = \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Notation.

$\mathbb{R}^n$: set of all n-vectors with real components. $x_i \in \mathbb{R}$, $i=1,\cdots,n$

$\mathbb{C}^n$: complex $x_i \in \mathbb{C}$, $i=1,\cdots,n$.

Vector eq.

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 = 1 \\ x_1 - x_3 = 0. \end{cases}$$

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{b}$$