Lec 20. Matrix representation of linear transformation

Part II.

Similarity

$$T: \mathbb{R}^n \to \mathbb{R}^n$$

$$\vec{x} \mapsto A\vec{x}$$

basis $B = \{\vec{b}_1, \cdots, \vec{b}_n\}$.

$$\begin{tikzcd}
T: & \mathbb{R}^n \arrow[r] & \mathbb{R}^n \arrow[d, "P_B^{-1}"] \\
\text{Coord.} & \mathbb{R}^n \arrow[u, "P_B"] \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

$$[T]_B = P_B^{-1} \, T \, P_B$$

$$V = [\vec{b_1}, \cdots, \vec{b_n}] \in \mathbb{R}^{n \times n}$$

$$[T]_B = V^{-1} A V \in \mathbb{R}^{n \times n}.$$

What if $\vec{b_1}, \cdots, \vec{b_n}$ are eigenvectors of A?

$$A \vec{b_i} = \lambda_i \vec{b_i}, \quad i = 1, \cdots, n.$$

$$[T]_B = V^{-1} A V = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

From the perspective of matrix representation of lin. trans. diagonalization means picking a basis B. s.t. the mat. rep. w.r.t. B is a diagonal matrix.

Ex. $T: \mathbb{P}_2 \to \mathbb{P}_2$.

$$[T(p)](x) = (x+1)\frac{dp}{dx}(x)$$

Is there a basis of $\mathbb{P}_2$ s.t.

$[T]_B$ is a diagonal matrix?

Sol 1 First pick a basis of $\mathbb{P}_2$.

$$E = \{1, x, x^2\}$$

Write down mat. rep. of T w.r.t. E.

$$[T]_E = \left[[T(1)]_E \ [T(x)]_E \ [T(x^2)]_E \right] \in \mathbb{R}^{3\times 3}.$$

$$T(1) = 0 \Rightarrow [T(1)]_E = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = x+1 \Rightarrow [T(x)]_E = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(x^2) = 2x(x+1) \Rightarrow [T(x^2)]_E = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$[T]_E = A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

② Diagonalize A.

$$0 = \begin{vmatrix} -\lambda & 1 & 0 \\ & 1-\lambda & 2 \\ & & 2-\lambda \end{vmatrix} = (-\lambda)(1-\lambda)(2-\lambda)$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

$$\lambda_1 = 0: \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \vec{V_1} = \vec{0}, \quad \vec{V_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1. \quad \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{V_2} = \vec{0}, \quad \vec{V_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2 \quad \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \vec{V_3} = \vec{0}. \quad \vec{V_3} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

③ $\vec{b_1} = 1 \cdot 1 + x \cdot 0 + x^2 \cdot 0 = 1.$

$\vec{b_2} = 1 \cdot 1 + x \cdot 1 + x^2 \cdot 0 = 1+x$

$\vec{b_3} = 1 \cdot 1 + x \cdot 2 + x^2 \cdot 1 = (1+x)^2$

Claim: $\mathcal{B} = \{\vec{b_1}, \vec{b_2}, \vec{b_3}\}.$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$\square.$

Verification step:

$$[T]_B = \left[[T(\vec{b_1})]_B \; [T(\vec{b_2})]_B \; [T(\vec{b_3})]_B \right]$$

$T(\vec{b_1}) = 0.$ $\quad [T(\vec{b_1})]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(\vec{b_2}) = 1+x.$ $\quad [T(\vec{b_2})]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$T(\vec{b_3}) = 2(1+x)^2.$ $\quad [T(\vec{b_3})]_B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ $\quad \square$

Similarity

<u>Def</u> $A, B \in \mathbb{R}^{n \times n}$ A is similar to B if there exists an invertible matrix V s.t.

$$A = V^{-1} B V.$$

For example. A is diagonalizable.

A is similar to D ← containing eigenvalues of A.

From perspective of mat. rep?

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "B"] & \mathbb{R}^n \arrow[d, "V^{-1}"] \\ \mathbb{R}^n \arrow[u, "V"] \arrow[r, "A"] & \mathbb{R}^n \end{tikzcd}$$

V

$[B]_V = V^{-1} B V$

A, B are similar means that they only differ by a change of basis.

<u>Thm</u>. $A, B \in \mathbb{R}^{n \times n}$ are similar

then $|A - \lambda I| = |B - \lambda I|$

Therefore. A, B share the same set of eigenvalues (counting multiplicity)

Pf: $A = V^{-1} B V$

$$\Rightarrow |A - \lambda I| = |V^{-1}(B - \lambda I) V|$$

$$= |V^{-1}| \cdot |B - \lambda I| \cdot |V|$$

$$= |V^{-1}| \cdot |V| \cdot |B - \lambda I|$$

$$= |V^{-1} \cdot V| \, |B - \lambda I|$$

$$= |B - \lambda I|$$

$\square$