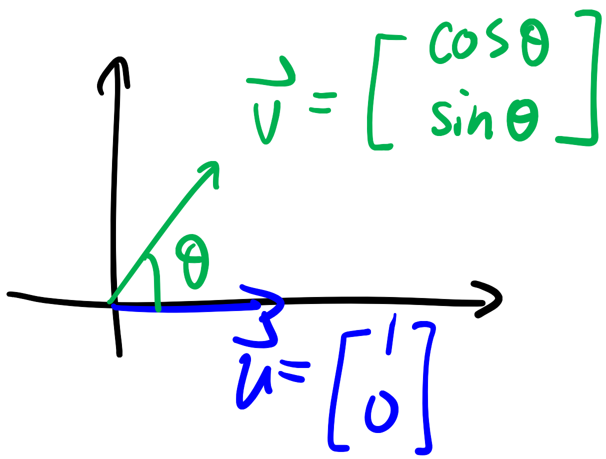
Lec 21 Geometry

Length & angles in $\mathbb{R}^n$

$\mathbb{R}^2$



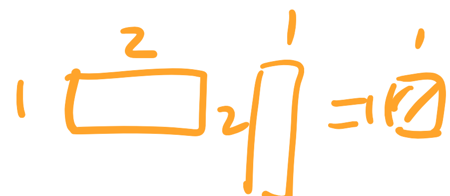
$\|\vec{v}\| := \sqrt{v_1^2 + v_2^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$

length

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 = \cos\theta$$

$$\theta = \arccos(\vec{u} \cdot \vec{v})$$?

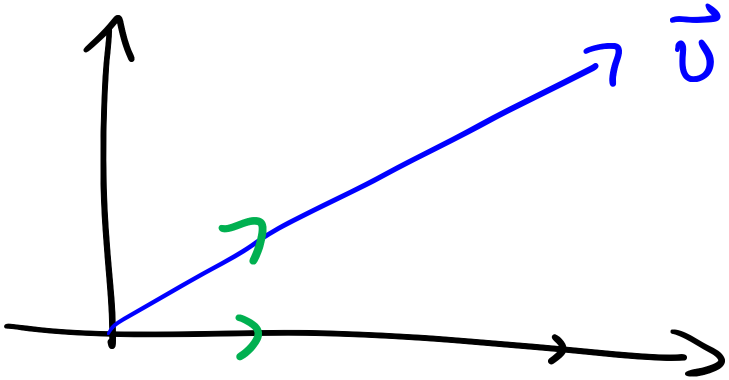
Def. $\vec{u}, \vec{v} \in \mathbb{R}^2$, the inner product of $\vec{u}, \vec{v}$ is



$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 = \vec{u}^T \vec{v} = [u_1 \ \ u_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The length of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2}$$



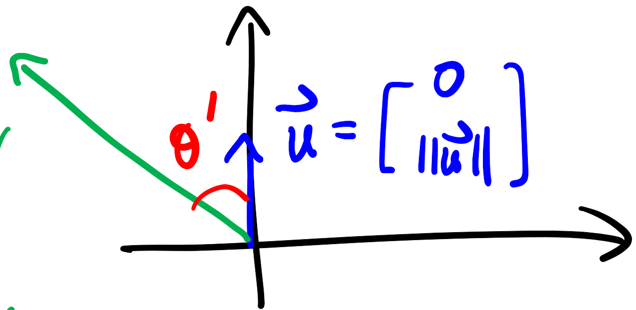
$$\vec{v} = \begin{bmatrix} \|\vec{v}\| \cos\theta \\ \|\vec{v}\| \sin\theta \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \|\vec{u}\| \\ 0 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \underbrace{\cos\theta}_{\text{angle in } \mathbb{R}^2.}$$

Is angle invariant w.r.t. rotation?

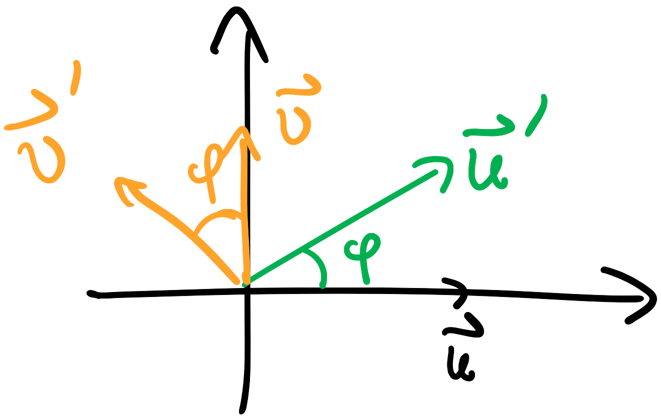
ex. rotate by $\frac{\pi}{2}$.



$$\vec{v} = \begin{bmatrix} \|\vec{v}\|(-\sin\theta) \\ \|\vec{v}\|\cos\theta \end{bmatrix}$$

$$\cos\theta' = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = \frac{\|\vec{u}\|\,\|\vec{v}\|\cos\theta}{\|\vec{u}\|\cdot\|\vec{v}\|} = \cos\theta.$$

Ex. rotation by any angle φ



$$\vec{u}' = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \vec{u}$$

standard matrix

$$\vec{v}' = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \vec{v}.$$

$$\cos\theta' = \frac{\vec{u}' \cdot \vec{v}'}{\|\vec{u}'\| \cdot \|\vec{v}'\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \cos\theta.$$

$$\vec{u}\cdot\vec{v}' = \vec{u}'^T\vec{v}' = \left(\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\vec{u}\right)^T \left(\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\vec{v}\right)$$

$$= \vec{u}^T \underbrace{\left(\begin{bmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\right)}_{I}\vec{v}$$

$$= \vec{u}^T\vec{v} = \vec{u}\cdot\vec{v}$$

Inner product is invariant to rotation!

$\mathbb{R}^2 \longrightarrow \mathbb{R}^n$

Geometric $\longrightarrow$ algebraic.

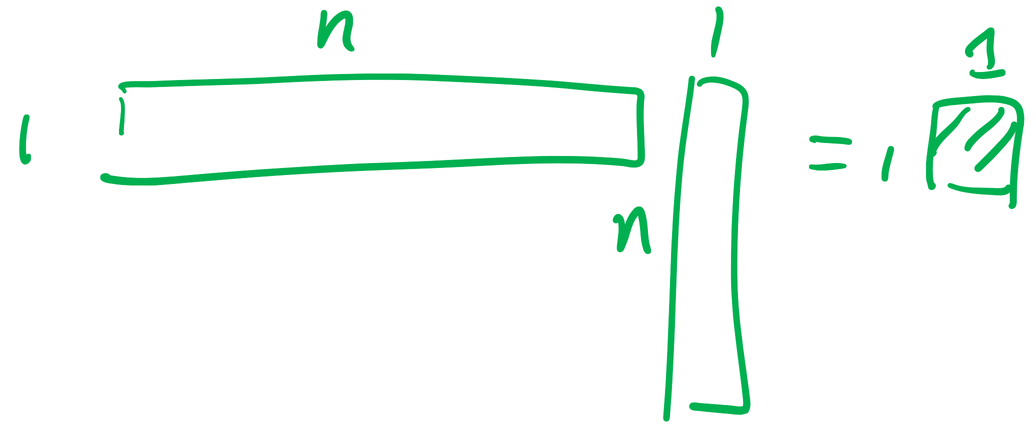
<u>Def</u> $\vec{u}, \vec{v} \in \mathbb{R}^n$. inner product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$$

Length of $\vec{v}$

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$$

Again $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$



Ex. $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. $\|\vec{v}\| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$

$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. $\|\vec{u}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1-1+0}{\sqrt{6}\cdot\sqrt{2}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Properties of inner product

1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

3) $c \in \mathbb{R}.\ (c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$

4) $\vec{u} \cdot \vec{u} \geq 0$, $\quad \vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 0 \Leftrightarrow \vec{u} = \vec{0}$

Pf: $\|\vec{u}\|^2 = u_1^2 + \cdots + u_n^2 = 0$

$\Leftrightarrow u_1 = \cdots = u_n = 0 \Leftrightarrow \vec{u} = \vec{0}$

Def Unit vector. $\|\vec{u}\| = 1$.

For any nonzero vector $\vec{u}$

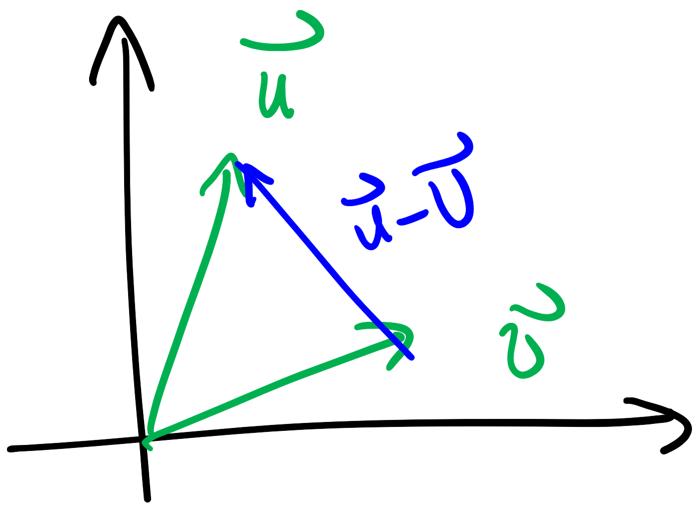
$$\vec{v} = \frac{1}{\|\vec{u}\|} \vec{u}$$ ← normalization

$$\vec{v} \cdot \vec{v} = \frac{1}{\|\vec{u}\|^2} \vec{u} \cdot \vec{u} = 1. \Rightarrow \|\vec{v}\| = 1.$$

Ex. $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \vec{v} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Def Distance between $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|$$



$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

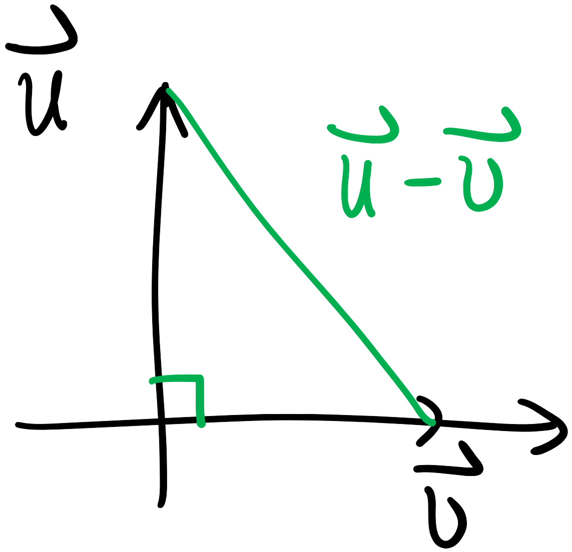
$$= \vec{u}\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

Special case $\vec{u}\cdot\vec{v}=0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$



Generalization of Pythagorean theorem in $\mathbb{R}^n$.

<u>Def</u> $\vec{u}, \vec{v} \in \mathbb{R}^n$. $\vec{u} \cdot \vec{v} = 0$

then $\vec{u}, \vec{v}$ are orthogonal vectors.

$\vec{u} \perp \vec{v}$

Ex. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{v}_1 \cdot \vec{v}_2 = -1 + 2 + 0 - 1 = 0 \Rightarrow \vec{v}_1 \perp \vec{v}_2$$

$$\vec{v}_1 \cdot \vec{v}_3 = 0$$