# Lec 22. Orthogonal systems

# Projection

Inner product. $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i \quad , \quad \|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0 \iff \theta = \frac{\pi}{2} \quad (\vec{u}, \vec{v} \neq \vec{0})$$

In Chap 4.

$\mathbb{R}^{2 \text{ or } 3} \longrightarrow \mathbb{R}^n \longrightarrow V$ 

In Chap 6.

equip vector space $V$ by an additional structure (inner product) to describe geometry.

$V$

+ equip

(regular) vector space. So far $\mathbb{R}^n$

Inner product

$=$ 

Inner product Space.

**Def** $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is orthogonal set

If $\vec{v}_i \in \mathbb{R}^n$, $\vec{v}_i \neq \vec{0}$,

$\vec{v}_i \perp \vec{v}_j$ for all $1 \leq i \neq j \leq k$.

Orthonormal set

1) orthogonal set

2) $\|\vec{v}_i\| = 1$, $\forall\ 1 \leq i \leq k$.

Ex. $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

1) orthogonal set? ✓

$\vec{v}_1 \cdot \vec{v}_2 = 0$, $\vec{v}_1 \cdot \vec{v}_3 = 0$, $\vec{v}_2 \cdot \vec{v}_3 = -1-4+5=0$

2) orthonormal set? ✗

$\|\vec{v}_1\| = \sqrt{4+1} = \sqrt{5}$.

$\vec{w}_1 = \frac{1}{\sqrt{5}} \vec{v}_1$, $\vec{w}_2 = \frac{1}{\sqrt{6}} \vec{v}_2$, $\vec{w}_3 = \frac{1}{\sqrt{30}} \vec{v}_3$

orthonormal set!

**Thm** $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is an orthogonal set

then lin. indep.

**Pf.** $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$

Pick any $\vec{v}_i$, $1 \le i \le k$.

$$\vec{v}_i \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) = 0$$

Orthogonal set. $\vec{v}_i \cdot \vec{v}_j = 0$, for all $i \neq j$.

$$c_i \underbrace{\vec{v}_i \cdot \vec{v}_i}_{} \overset{\neq 0}{} = 0 \implies c_i = 0 \qquad \square$$

Def $\{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq \mathbb{R}^n$ is an orthogonal basis (OB) of subspace $W \subseteq \mathbb{R}^n$

1) Orthogonal set

2) basis $\rightarrow \{\vec{v}_1, \cdots, \vec{v}_k\} \subseteq W$

$\dim W = k$.

If orthonormal set $\Rightarrow$ orthonormal basis (ONB)

Why OB/ONB is useful?

Take $W = \mathbb{R}^n$.

$\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ is a basis

$\vec{v} \in \mathbb{R}^n$. $[\vec{v}]_{\mathcal{B}}$.

Old method: solve a lin. sys.

$$\left[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_n \,\middle|\, \vec{v}\right] \longrightarrow [\vec{v}]_B .$$

What if B is OB?

$$[\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{v} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n$$

$$\vec{b}_i \cdot \vec{v} = \vec{b}_i \cdot \left(c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n\right) = c_i\, \vec{b}_i \cdot \vec{b}_i$$

$$\Rightarrow \quad c_i = \frac{\vec{b_i} \cdot \vec{v}}{\vec{b_i} \cdot \vec{b_i}} \qquad 1 \le i \le n.$$

If ONB $\quad \vec{b_i} \cdot \vec{b_i} = 1$

$$c_i = \vec{b_i} \cdot \vec{v}$$

Ex. $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

OB of $\mathbb{R}^3$ $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find $[\vec{u}]_{\mathcal{B}}$.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{u}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{2}{5}$$

$$c_3 = \frac{-1+5}{30} = \frac{2}{15}.$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{u}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{2}{6} = \frac{1}{3}.$$

Spoiler: set of arbitrary vectors.

$\xrightarrow{\text{reduce}}$ Orthogonal set?

Gram-Schmidt process.

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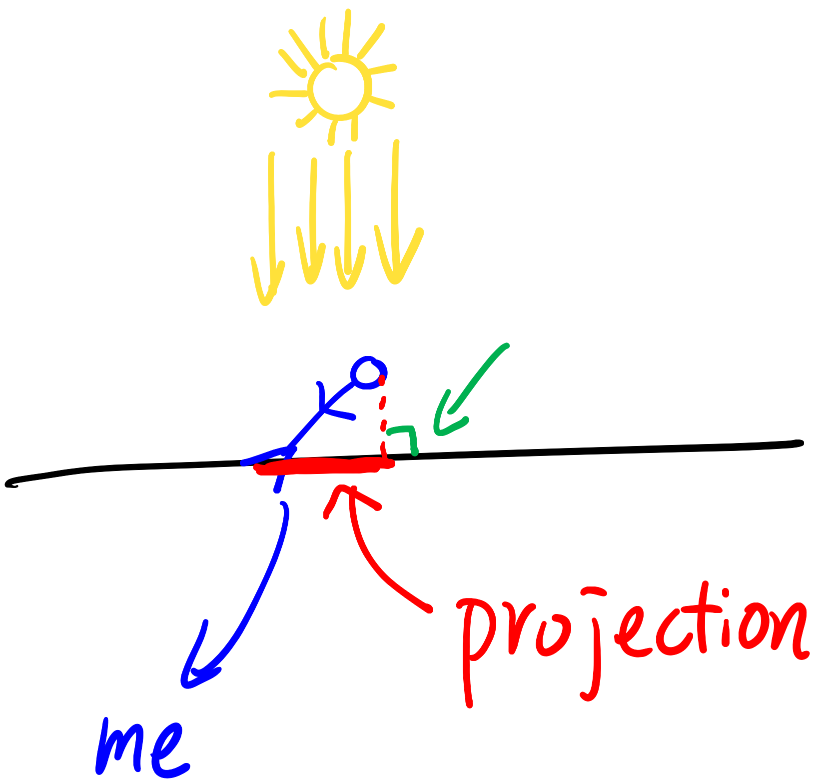
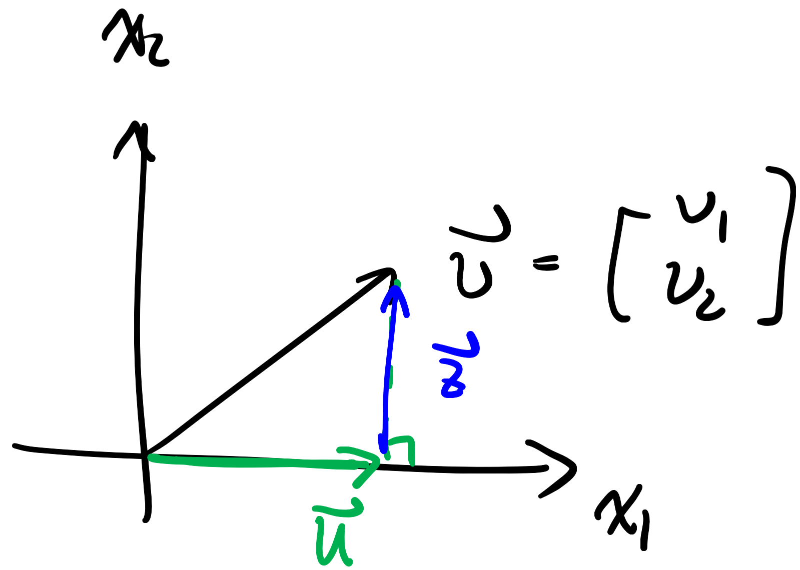
We need to first learn:

# Projection




Projection of $\vec{v}$ onto the $x_1$-axis

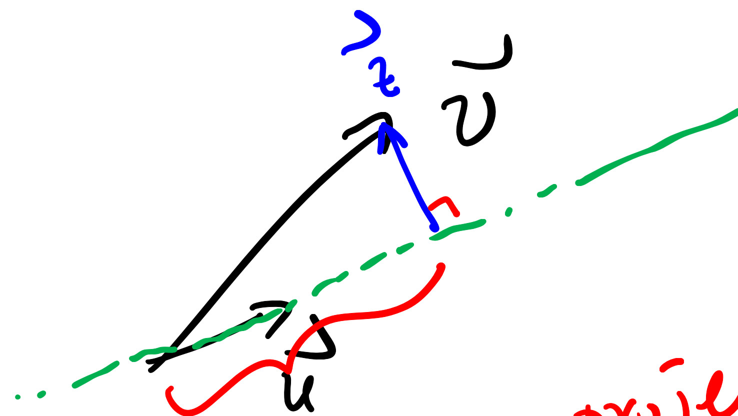
$$\vec{u} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \qquad \vec{z} = \vec{v} - \vec{u} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \qquad \vec{u} \cdot \vec{z} = 0.$$

$$\vec{u} \perp \vec{z}$$

Ex. $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v} = \alpha\vec{u} + \vec{z} \qquad \vec{u} \perp \vec{z}$$

$$\begin{array}{c} \alpha\vec{u} \\ \cap \\ \mathrm{Span}\{\vec{u}\} \end{array}$$




$\alpha\vec{u} \longrightarrow$ projection.

$$\vec{u} \cdot \vec{v} = \vec{u} \cdot (\alpha\vec{u} + \vec{z}) = \alpha\, \vec{u} \cdot \vec{u}$$

$$\alpha = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} = \frac{3}{5}$$

$$\vec{z} = \vec{v} - \alpha\vec{u} = \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

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In the example above.

$$W = \operatorname{span}\{\vec{u}\}. \qquad \dim W = 1.$$

Decompose.

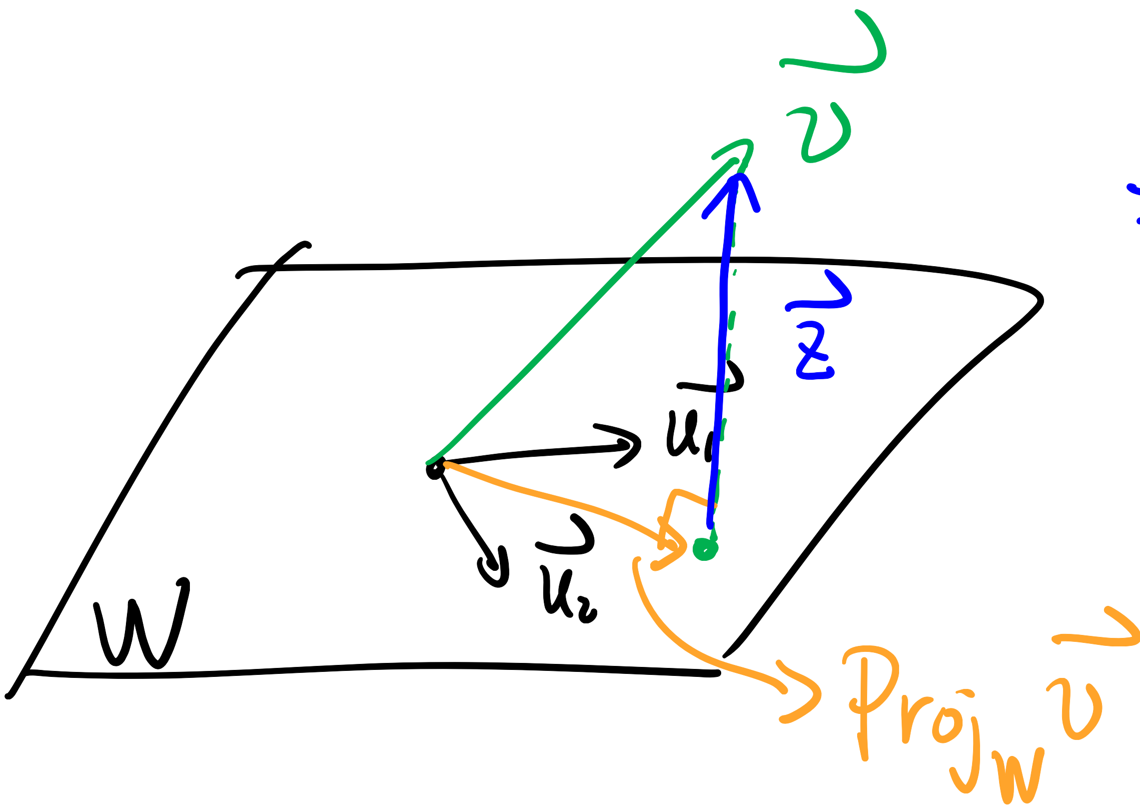
$$\vec{v} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}}_{\in W} + \vec{z}$$

$$\vec{z} \perp W$$

$$\mathrm{proj}_W \vec{v}$$

$$\Rightarrow \quad \vec{v} = \mathrm{proj}_W \vec{v} + \vec{z}$$

$$W = \text{span}\underbrace{\{\vec{u}_1, \cdots, \vec{u}_k\}}_{\text{ONB}} \quad \dim W = k.$$




$$\vec{z} \perp W$$
$$\Updownarrow$$
$$\vec{z} \cdot \vec{u}_i = 0$$
$$1 \le i \le k.$$

$$\vec{v} = \underbrace{\alpha_1 \vec{u}_1 + \cdots + \alpha_k \vec{u}_k}_{W} + \vec{z}$$

unity assumes ONB

$$\boxed{\vec{u}_i \cdot \vec{v} = \alpha_i}$$

$$\Rightarrow \operatorname{Proj}_W \vec{v} = (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + \cdots + (\vec{u}_k \cdot \vec{v})\vec{u}_k$$