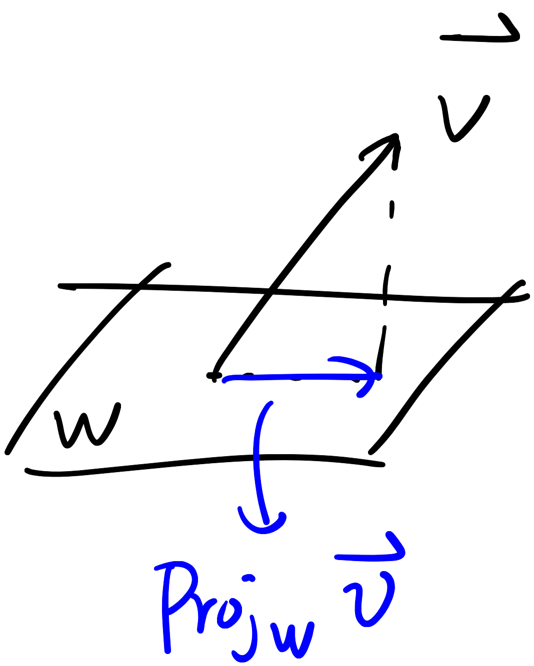
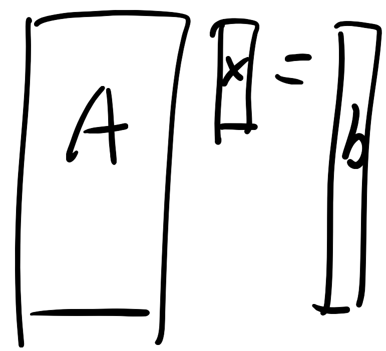
Lec 23. Projection.

least squares problem



"$A\vec{x} = \vec{b}$"



Col(A)

Ex. $A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{orthogonal set}}$ $\quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Compute $\text{Proj}_{\text{Col}(A)} \vec{b}$

$$= \frac{\vec{a}_1 \cdot \vec{b}}{\vec{a}_1 \cdot \vec{a}_1} \vec{a}_1 + \frac{\vec{a}_2 \cdot \vec{b}}{\vec{a}_2 \cdot \vec{a}_2} \vec{a}_2$$

$$= \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{0}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$U \in \mathbb{R}^{n \times k}$.

$U = [\vec{u}_1 \cdots \vec{u}_k]$ $\quad U$ has orthonormal columns

if $\quad \vec{u}_i \cdot \vec{u}_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$

i.e., $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is an orthonormal set.

$$(\underbrace{U^T U}_{\in \mathbb{R}^{k \times k}})_{ij} = \vec{u}_i^{\,T} \vec{u}_j = \vec{u}_i \cdot \vec{u}_j$$

$$U^T U = I_k.$$

Special case: $k=n$.

$$U^T U = I_n. \quad U \in \mathbb{R}^{n\times n}.$$

$\{\vec{u}_1, \dots, \vec{u}_n\}$ is ONB of $\mathbb{R}^n$.

$$U^{-1} = U^T.$$ special when $k=n$.

$$\begin{array}{cc} & k \\ n & \boxed{U} \end{array} : \quad \boxed{U^T}\,\boxed{U} = \boxed{I_k}$$

$$\underbrace{\boxed{U}\,\boxed{U^T}}_{\text{projection.}}$$

$$\begin{array}{cc} & n \\ n & \boxed{U} \end{array} \qquad UU^T = UU^{-1} = I_n .$$

$U \in \mathbb{R}^{n \times n}$. ONB.

orthogonal matrix

$\Updownarrow$ orthonormal "

Confusing notation!

Connect to projection.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$U = [\vec{u}_1, \vec{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$\operatorname{Proj}_{\operatorname{Col}(A)} \vec{b} = \vec{u}_1 (\vec{u}_1 \cdot \vec{b}) + \vec{u}_2 (\vec{u}_2 \cdot \vec{b})$$

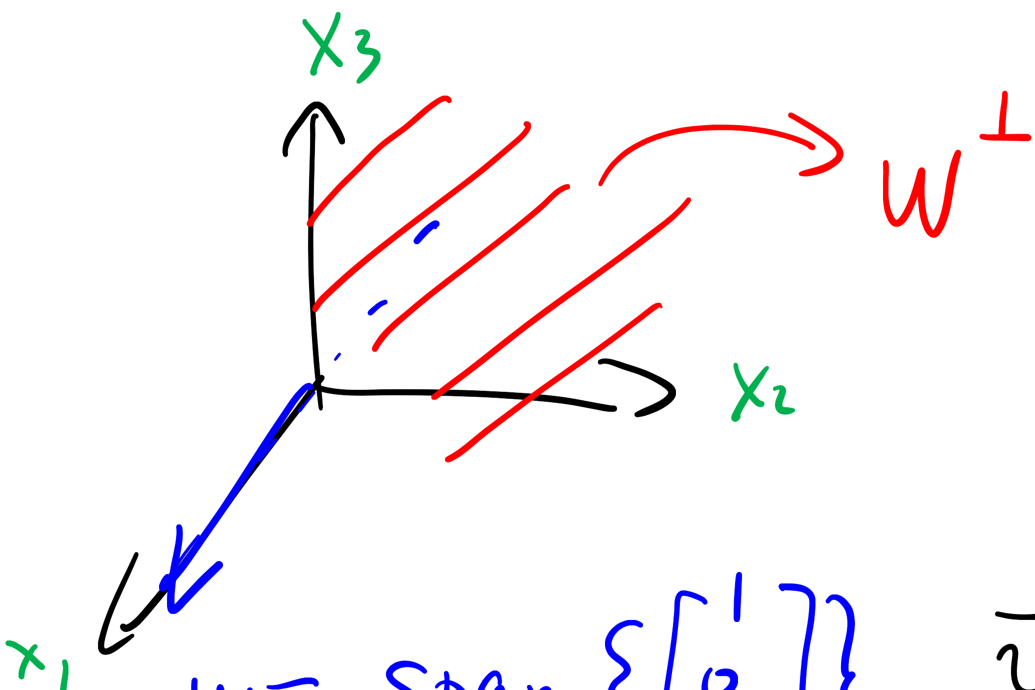
$$= (\vec{u}_1 \vec{u}_1^{\,T} + \vec{u}_2 \vec{u}_2^{\,T}) \vec{b}$$

$$= \underbrace{U U^T}_{} \vec{b}$$

$$\left(U U^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \vec{u}_1^{\,T} \\ \vec{u}_2^{\,T} \end{bmatrix} = \vec{u}_1 \vec{u}_1^{\,T} + \vec{u}_2 \vec{u}_2^{\,T} \right)$$

Def $W \subseteq \mathbb{R}^n$ is a subspace.

$$W^\perp = \{ \vec{v} \in \mathbb{R}^n \mid \vec{v} \perp \vec{w} \text{ for all } \vec{w} \in W \}$$



$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\quad \vec{v} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \Leftrightarrow v_1 = 0.$

Thm. 1) $W^\perp$ is a subspace

2) $(W^\perp)^\perp = W$

Thm. (Uniqueness of projection)

$W \subseteq \mathbb{R}^n$ is a subspace. any vector $\vec{v} \in \mathbb{R}^n$ has a unique decomposition

$$\vec{v} = \vec{w} + \vec{z}, \quad \vec{w} \in W, \ \vec{z} \in W^\perp.$$

$$\left(\vec{w} = \mathrm{Proj}_W \vec{v} \right)$$

Pf: (of uniqueness).

$$\vec{v} = \vec{w} + \vec{z} = \vec{w}' + \vec{z}'$$

$$\vec{w}, \vec{w}' \in W$$

$$\vec{z}, \vec{z}' \in W^\perp$$

$$\Rightarrow \quad \vec{w} - \vec{w}' = \vec{z}' - \vec{z}$$

$$\underbrace{(\vec{w}-\vec{w}')\cdot(\vec{w}-\vec{w}')}_{\|\vec{w}-\vec{w}'\|^2} = 0$$

$\Rightarrow \vec{w} = \vec{w}'$, $\vec{z} = \vec{z}'$. Therefore uniqueness

$\square$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ Find $\mathrm{Col}(A)^{\perp}$

$$\mathrm{Col}(A)^{\perp} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; \vec{a}_1 \cdot \vec{x} = 0,\ \vec{a}_2 \cdot \vec{x} = 0 \right\}$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \implies \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$\Rightarrow$ sol set is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} c \;\middle|\; c \in \mathbb{R} \right\}$,

$$= \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

More generally.

$$A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_k \end{bmatrix} \qquad \mathrm{Col}(A)^{\perp}$$

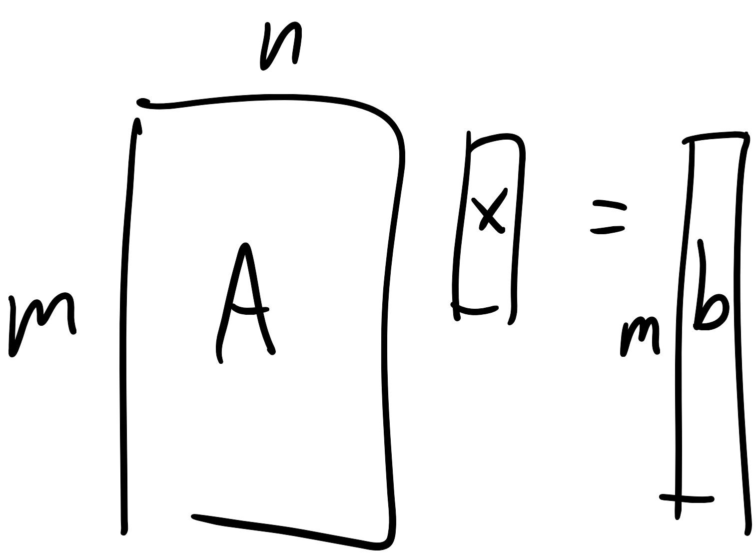
$$\begin{cases} \vec{a}_1 \cdot \vec{x} = 0 \\ \vdots \\ \vec{a}_k \cdot \vec{x} = 0 \end{cases} \Rightarrow \begin{cases} \vec{a}_1^{T} \vec{x} = 0 \\ \vdots \\ \vec{a}_k \vec{x} = 0 \end{cases} \Rightarrow A^{T} \vec{x} = \vec{0}$$

$$\mathrm{Col}(A)^{\perp} = \mathrm{Nul}(A^{T})$$

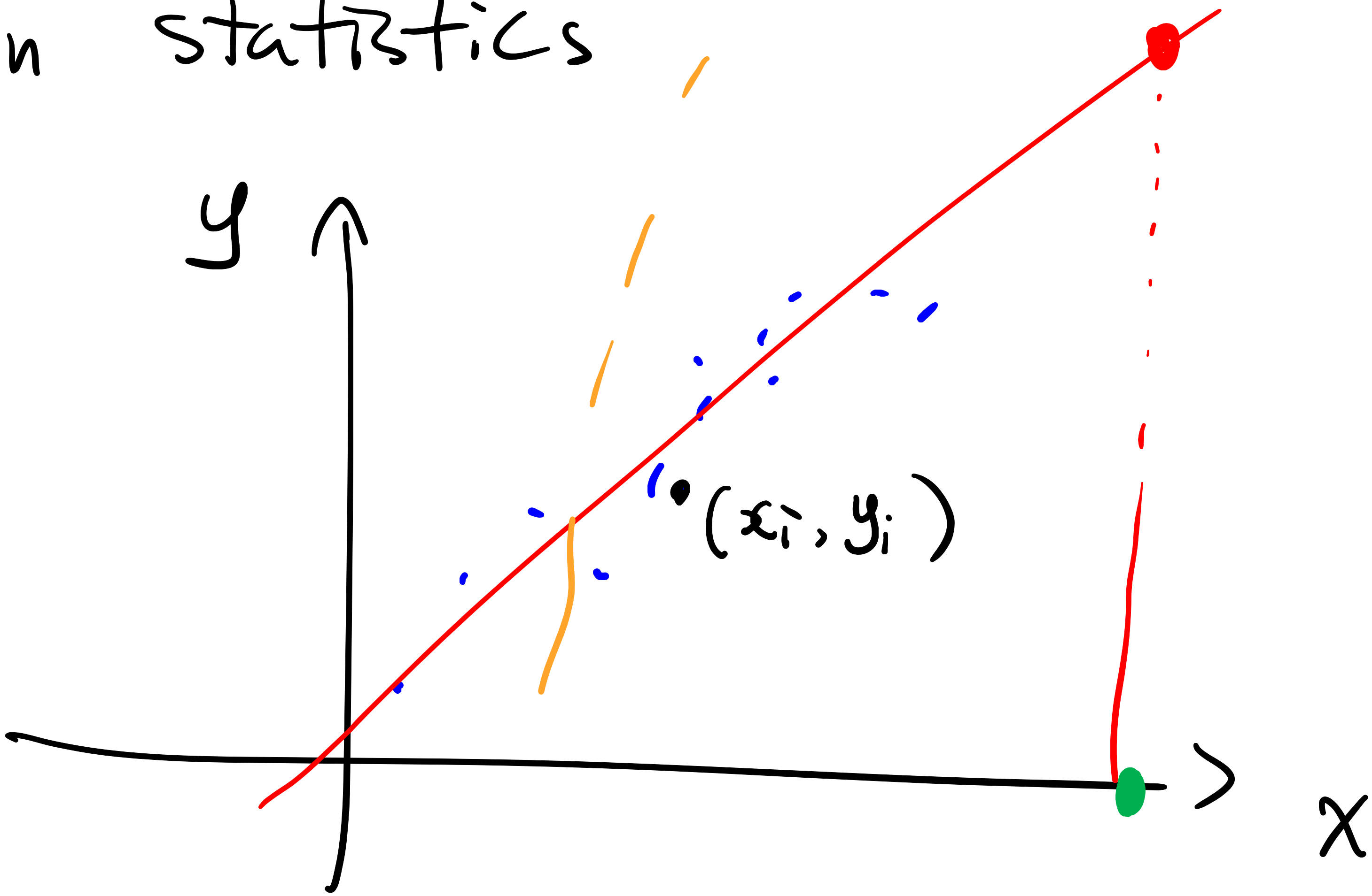
Least squares

$A\vec{x} = \vec{b}$ has no sol.

$A \in \mathbb{R}^{m \times n}$, $m > n$



In statistics



observations (x_i, y_i)

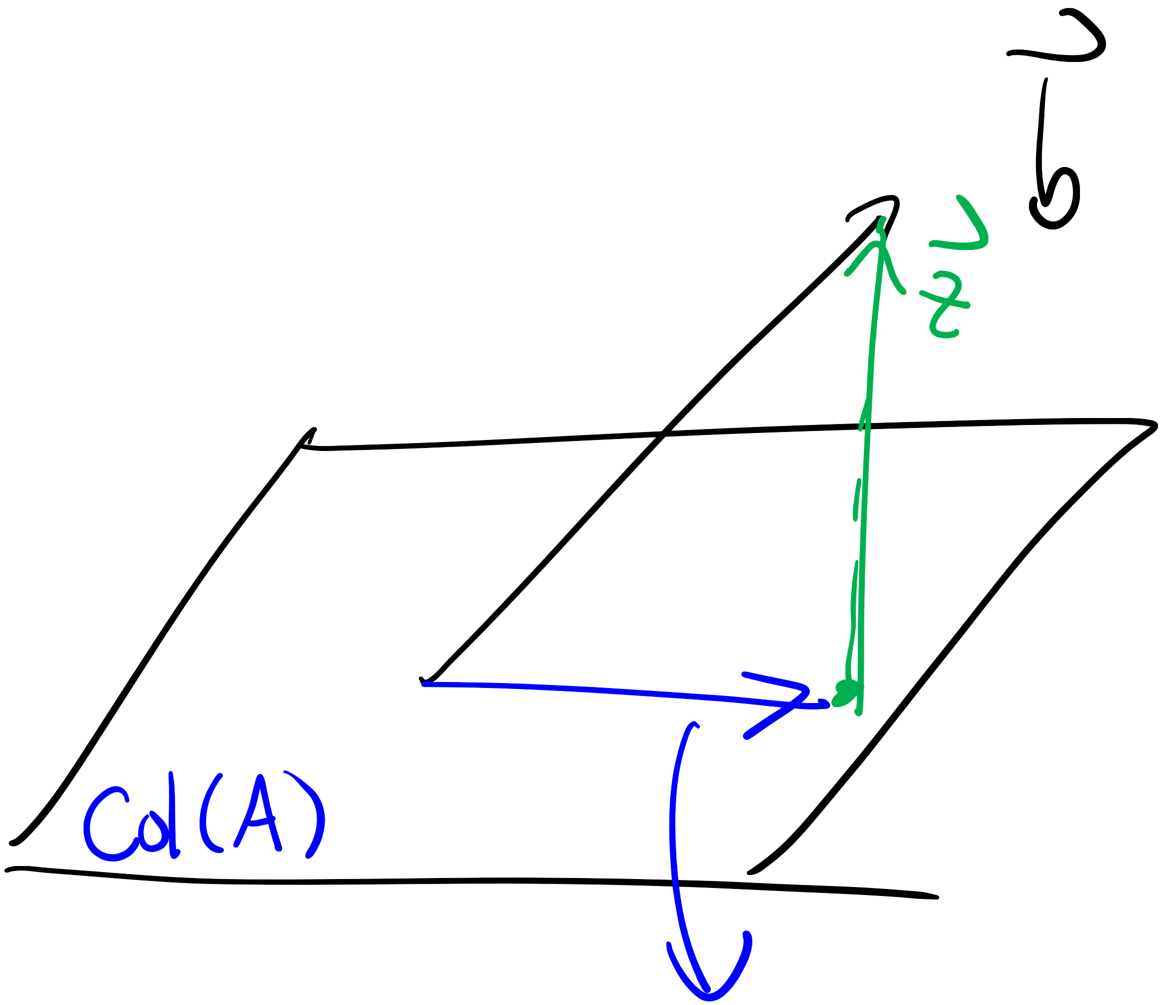
$$a x_i + b = y_i, \quad i = 1, \cdots, m$$

unknowns are a, b.

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_m & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\Longleftrightarrow \quad \min_{\vec{x}} \| \underbrace{A\vec{x} - \vec{b}}_{\text{residual}} \|^2$$



$$\text{Proj}_{\text{Col}(A)} \vec{b} = A\vec{x}$$

minimal distance $\|\vec{z}\| = \|\vec{b} - \text{Proj}_{\text{Col}(A)}\vec{b}\|$.