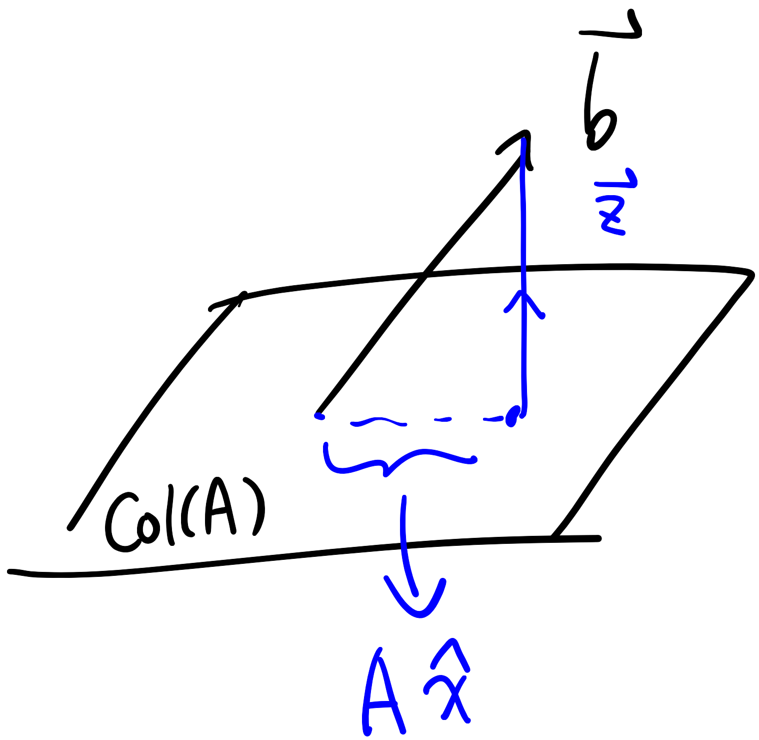
Lec 24. Least squares

Gram Schmidt



$A\hat{x} \in \mathrm{col}(A)$

$$\vec{b} - A\hat{x} \in \underbrace{\mathrm{col}(A)^{\perp}}_{= \, Nul(A^T)}$$

$$\Longleftrightarrow \quad A^T(\vec{b} - A\hat{x}) = 0$$

$$\Leftrightarrow \quad \boxed{A^T A \hat{x} = A^T \vec{b}} \rightarrow$$ ALWAYS has a sol.

sol to least squares problem

Normal equation.

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} A \end{bmatrix} \end{array} \; \begin{array}{cc} & 1 \\ n & \begin{bmatrix} x \end{bmatrix} \end{array} = \begin{array}{cc} & 1 \\ m & \begin{bmatrix} b \end{bmatrix} \end{array} \qquad \begin{array}{cc} & n \\ n & \begin{bmatrix} A^T A \end{bmatrix} \end{array} \begin{bmatrix} \hat{x} \end{bmatrix} = \begin{array}{cc} & 1 \\ n & \begin{bmatrix} A^T b \end{bmatrix} \end{array}$$

No sol!

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

Find least square sol.

$$A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right] \longrightarrow \hat{x} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$$

$A\hat{x} = -\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

minimal distance

$\|\vec{b} - A\hat{x}\|$.

$$y_i = b\, e^{a x_i}$$

x_i : date

y_i : # infections

$$\ln y_i = a x_i + \underbrace{\ln b}_{\tilde{b}}$$

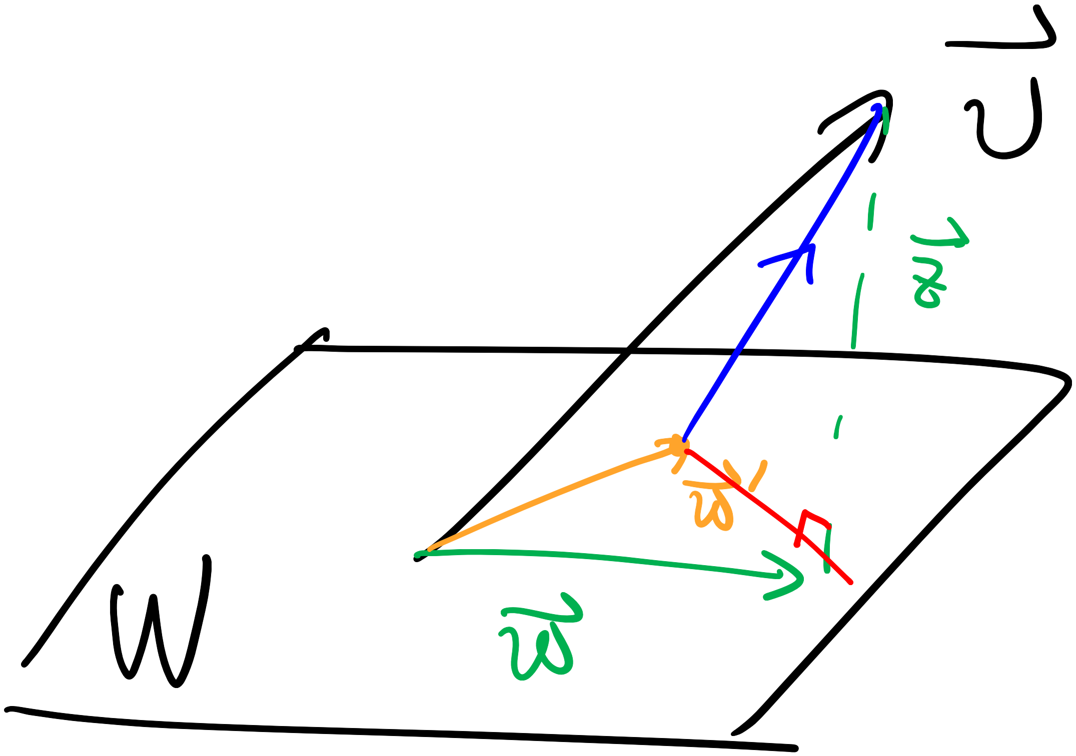
Thm (Best approximation)

$W \subseteq \mathbb{R}^n$ subspace. $\vec{v} \in \mathbb{R}^n$

$\vec{v} = \vec{w} + \vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^\perp$

$\vec{w} = \mathrm{Proj}_W \vec{v}$

$\|\vec{z}\| \leq \|\vec{v} - \vec{w}'\|$ for ANY $\vec{w}' \in W$



Pf: For any $\vec{w}' \in W$

$$\| \vec{v} - \vec{w}' \|^2 = \| \underbrace{(\vec{v} - \vec{w})}_{\in W^{\perp}} + \underbrace{(\vec{w} - \vec{w}')}_{\in W} \|^2$$

$$= \|\vec{v}-\vec{w}\|^2 + \|\vec{w}-\vec{w}'\|^2 \quad \text{Pythagorean thm.}$$

$$\geq \|\vec{z}\|^2 \qquad \square .$$

Gram-Schmidt process

$\{\vec{v}_1, \cdots, \vec{v}_k\}$ $\quad \vec{v}_i \in \mathbb{R}^n$ lin. indep.

generate an orthogonal set

$\{\vec{w}_1, \cdots, \vec{w}_k\}$.

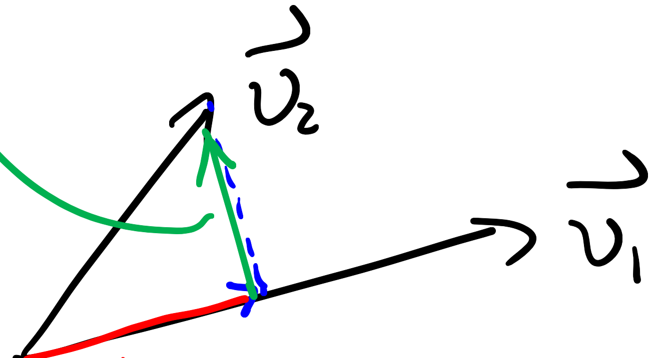
s.t.

$\text{Span}\{\vec{w}_1, \cdots, \vec{w}_k\} = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_k\}$

$\vec{w}_1 = \vec{v}_1$. $\qquad W_1 = \text{span}\{\vec{w}_1\} = \text{span}\{\vec{v}_1\}$.

$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{W_1} \vec{v}_2 \qquad W_2 = \text{span}\{\vec{w}_1, \vec{w}_2\}$

$\qquad\qquad\qquad\qquad\qquad\qquad = \text{span}\{\vec{v}_1, \vec{v}_2\}$.



remove

$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{W_2} \vec{v}_3 \qquad \cdots$

Ex. Find an OB for the sol set of

$$x_1 + 2x_2 + x_3 + 2x_4 = 0.$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 0 \end{array}\right]$$

1) Find a basis

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

sol set is $\{ c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \mid c_1, c_2, c_3 \in \mathbb{R} \}$

2) Apply G-S.

$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \mathrm{Proj}_{\mathrm{span}\{\vec{w}_1\}} \vec{v}_1 = \vec{v}_2 - \frac{\vec{w}_1 \cdot \vec{v}_2}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - \operatorname{Proj}_{\operatorname{span}\{\vec{w}_1, \vec{w}_2\}} \vec{v}_3$$

$$= \vec{v}_3 - \frac{\vec{w}_1 \cdot \vec{v}_3}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 - \frac{\vec{w}_2 \cdot \vec{v}_3}{\vec{w}_2 \cdot \vec{w}_2} \vec{w}_2$$

$$= \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is OB of col set.