Lec 25.

General inner product spaces.

$$\mathbb{R}^n \longrightarrow V$$

$$\vec{u}\cdot\vec{v} \qquad\qquad \langle \vec{u}, \vec{v} \rangle$$

Inner product space $\mathbb{R}^N$

(General) inner product space.

Def Vector space V. An inner product is a function that maps a pair of vectors $\vec{u}, \vec{v} \in V$ to a real number

denote this function by $\langle \vec{u}, \vec{v} \rangle$

① $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

② $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

③ $c \in \mathbb{R}$, $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

④ $\langle \vec{u}, \vec{u} \rangle \geq 0$, $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$

Ex. $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $\vec{u}, \vec{v} \in \mathbb{R}^2$

$$\langle \vec{u}, \vec{v} \rangle := \vec{u}^T A \vec{v} = a u_1 v_1 + b u_2 v_2$$

Is it an inner prod on $\mathbb{R}^2$?

False. $a = b = 0$. $\langle \vec{u}, \vec{u} \rangle \equiv 0 \quad \forall \vec{u} \in \mathbb{R}^2$.

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other cases: 1) $a > 0$, $b = 0$. $\langle \vec{u}, \vec{v} \rangle = a u_1 v_1$

$\times$ $\langle \vec{u}, \vec{u} \rangle = a u_1^2 = 0 \Rightarrow u_1 = 0$. $u_2 \in \mathbb{R}$.

2) $a > 0$, $b > 0$. $\langle \vec{u}, \vec{u} \rangle = a u_1^2 + b u_2^2 = 0 \Rightarrow u_1 = 0, u_2 = 0$

$\Rightarrow \vec{u} = \vec{0}$ ✓ inner product

3) $a>0,\ b<0 \qquad \langle \vec{u},\vec{u}\rangle = au_1^2+bu_2^2=0$

$$\Rightarrow u_1^2 = -\underbrace{\frac{b}{a}}_{>0.} u_2^2 . \quad u_1 = \pm\sqrt{-\frac{b}{a}}\, u_2$$

$\times$

4) $a<0,\ b<0. \quad \langle \vec{u},\vec{u}\rangle = au_1^2+bu_2^2=0 \Rightarrow u_1=u_2=0.$

$\langle \vec{u},\vec{u}\rangle \le 0 \qquad \times$

Think: $A = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$

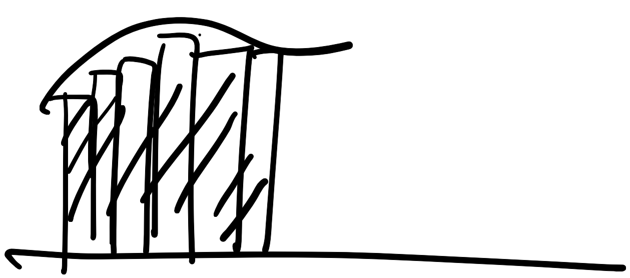
$\langle \vec{u},\vec{v}\rangle = \vec{u}^T A \vec{v}.$ Condition of A so that $\langle \vec{u},\vec{v}\rangle$ is an inner product?

Ex. $V = \mathbb{P}_n = \{\text{polynomials of degree} \leq n\}$.

$f, g \in V$.

$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$.

④ $\langle f, f \rangle = \int_0^1 f^2(x)\, dx = 0$



$\Rightarrow f(x) = 0$ on $[0,1]$

$\Leftrightarrow$ any $x \in [0,1]$ is a root of f

By the fundamental thm of algebra the only way f can have ∞ number of roots is $f = 0$.

$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$ is an inner prod.

Ex. $V = \mathbb{P}_n$.

$$\langle f, g \rangle = \int_0^1 f'(x)\, g'(x)\, dx .$$

Inner product ?

False. $f(x) \equiv 1$. $\langle 1, 1 \rangle = 0$.

$$\int_0^1 f'(x)^2\, dx = 0 \Rightarrow f'(x) = 0 \Rightarrow f(x) = c$$

for any c.

Ex. $V = \mathbb{P}_n$

$$\langle f, g \rangle = \int_0^1 f'(x)\, g'(x)\, dx + f(0)\, g(0)$$

$$\langle f, f \rangle = \underbrace{\int_0^1 f'^2(x)\, dx} + \underbrace{f^2(0)} = 0.$$

$$\Rightarrow \begin{cases} \int_0^1 f'^2(x)\, dx = 0 \\ f(0) = 0 \end{cases} \quad \Rightarrow f(x) = c \quad \Rightarrow f(x) = 0$$

✓ Inner product.

Everything in Chap 6

$\mathbb{R}^n \longrightarrow V \quad \langle \vec{u}, \vec{v} \rangle$

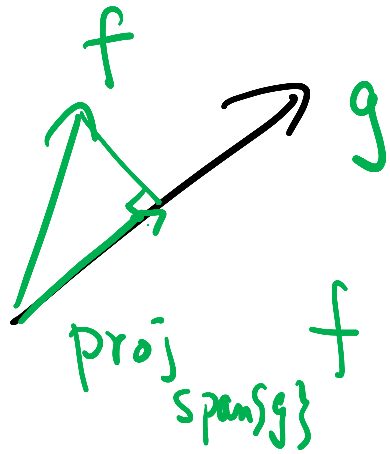
① length. $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

angle $\cos\theta = \dfrac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \, \|\vec{v}\|} \in [-1, 1]$

$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \cdot \|\vec{v}\|$ Cauchy-Schwarz inequality.

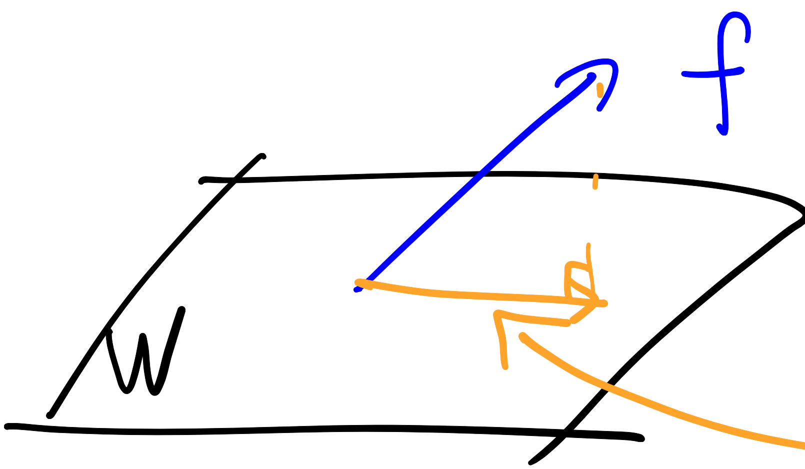
orthogonality $\vec{u} \perp \vec{v} \iff \langle \vec{u}, \vec{v} \rangle = 0$.

② Projection



Gram-Schmidt.

③



best approximation
least squares.

Ex. $P_1(x) = 1$, $P_2(x) = x$, $P_3(x) = \frac{3}{2}x^2$

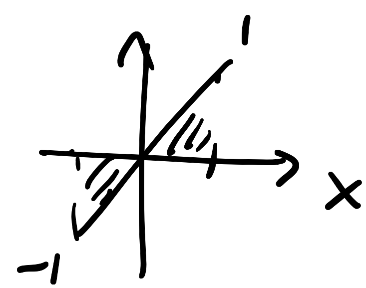
Perform Gram-Schmidt

Inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$

$$w_1(x) = P_1(x) = 1, \qquad \langle w_1, w_1 \rangle = \int_{-1}^{1} 1\, dx = 2$$

$$w_2(x) = P_2(x) - \frac{\langle w_1, P_2 \rangle}{\langle w_1, w_1 \rangle} w_1(x) \qquad \langle w_1, P_2 \rangle = \int_{-1}^{1} x\, dx = 0$$

$$= x$$



$$\langle w_2, w_2 \rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$w_3(x) = p_3(x) - \frac{\langle w_1, p_3 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle w_2, p_3 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= \frac{3}{2}x^2 - \frac{1}{2}$$

$$\left(\begin{aligned} \langle w_1, p_3 \rangle &= \int_{-1}^{1} 1 \cdot \frac{3}{2} x^2 dx = 1 \\ \langle w_2, p_3 \rangle &= \int_{-1}^{1} \frac{3}{2} x^3 dx = 0 \end{aligned} \right)$$

an orthogonal set is $\{1, x, \frac{3}{2}x^2 - \frac{1}{2}\}$

Q: an orthonormal set ?

(First 3) Legendre polynomial.

$\subseteq$ orthogonal polynomial.