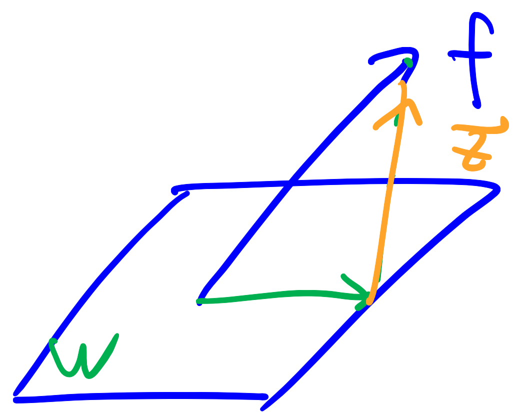
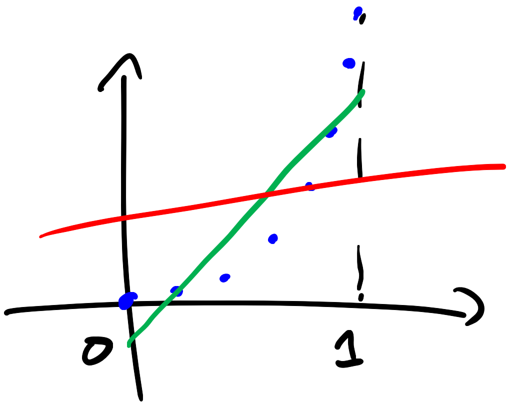
Lec 26. inner product space. II.

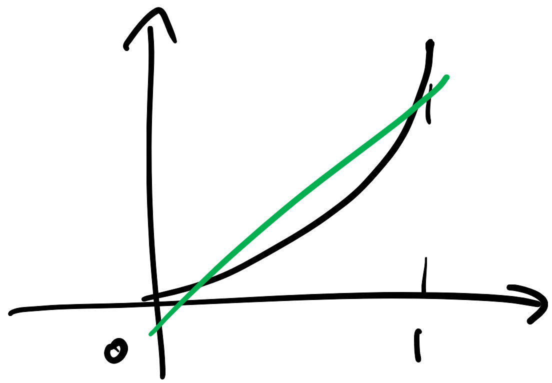
real symmetric matrix.

<u>Ex</u>. Find best approximation to $f(x)=x^2$ in span$\{1, x\}$ with respect to inner product

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$$







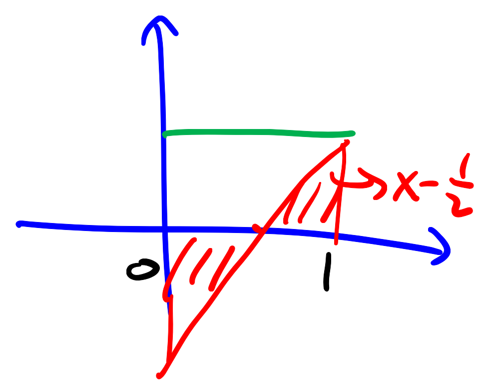
Answer: $\text{Proj}_{\text{span}\{1,x\}} f$

$\int_0^1 1 \cdot x \, dx = \frac{1}{2}$ not ortho.

Apply Gram-Schmidt.

$w_1(x) = 1$

$w_2(x) = x - \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{1}{2}$



$$\mathrm{Proj}_{\mathrm{span}\{w_1, w_2\}} f = \frac{\langle w_1, f\rangle}{\langle w_1, w_1\rangle} w_1 + \frac{\langle w_2, f\rangle}{\langle w_2, w_2\rangle} w_2$$

$$= \frac{\frac{1}{3}}{1} \cdot 1 + \frac{\frac{1}{12}}{\frac{1}{12}} \cdot \left(x - \frac{1}{2}\right)$$

$$= x - \frac{1}{6}$$

$$\begin{cases} \langle w_1, f\rangle = \int_0^1 1 \cdot x^2 \, dx = \frac{1}{3} \\ \langle w_2, f\rangle = \int_0^1 \left(x - \frac{1}{2}\right) x^2 \, dx = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{3} = + \frac{1}{12} \\ \langle w_1, w_1\rangle = 1 \\ \langle w_2, w_2\rangle = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = 2 \int_0^{\frac{1}{2}} x^2 \, dx = 2 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{12} \end{cases}$$

Check:

$$\vec{z} = x^2 - \left(x - \frac{1}{6}\right)$$

$$\langle \vec{z}, 1 \rangle = \int_0^1 x^2 - x + \frac{1}{6}\, dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$$

$$\langle \vec{z}, x \rangle = \int_0^1 x^3 - x^2 + \frac{x}{6}\, dx = \frac{1}{4} - \frac{1}{3} + \frac{1}{12} = 0.$$

Complex inner product space $\mathbb{C}$

$\vec{u} \in \mathbb{C}^n \qquad \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad u_i \in \mathbb{C}. \qquad \vec{v} \in \mathbb{C}^n$

How about

$$\vec{u} \cdot \vec{v} \equiv \langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i v_i \quad ?$$

$$\vec{u} \cdot \vec{u} = \sum_{i=1}^{n} u_i^2$$

<u>Example:</u> $n=2 \qquad \vec{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\vec{u} \cdot \vec{u} = 1 \cdot 1 + i \cdot i = 1 - 1 = 0$$

<u>Correct</u> generalization: $\overline{a+ib} = a-ib$.

$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} \overline{u_i}\, v_i$$

Same example: $\vec{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\quad \vec{u}\cdot\vec{u} = 1\cdot 1 + (-i)\cdot i = 2$

$$\vec{u}\cdot\vec{u} = \sum_{i=1}^{n} |u_i|^2 \geq 0.$$

If $\vec{u}\cdot\vec{u} = 0 \Rightarrow |u_i|^2 = 0 \Rightarrow u_i = 0 \Rightarrow \vec{u} = \vec{0}$.

In real case. $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{C}^n, \qquad \overline{\vec{u}} = \begin{bmatrix} \bar{u}_1 \\ \vdots \\ \bar{u}_n \end{bmatrix}$$

$$\vec{u}^T = [u_1 \cdots u_n] \qquad \vec{u}^* \equiv \overline{\vec{u}}^T = [\bar{u}_1 \cdots \bar{u}_n]$$

Hermitian conjugate.

$$\vec{u} \cdot \vec{v} = \vec{u}^* \vec{v}$$

Orthogonal matrix

$U \in \mathbb{R}^{n \times n}$ orthogonal $\iff U^T U = I_n$

$\iff U U^T = I_n$

$\iff U^{-1} = U^T$

Ex. $U = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$U^T U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\vec{u}\cdot\vec{v} = (U\vec{a})^T(U\vec{b}) = \vec{a}^T U^T U \vec{b} = \vec{a}\cdot\vec{b}$$

$\vec{u} = U\vec{a}$

$\vec{v} = U\vec{b}$

change of coord. using an orthogonal matrix $\Rightarrow$ keeps inner prod invariant.

Complex case:

$U \in \mathbb{C}^{n\times n}$. unitary $\Leftrightarrow U^*U = I$

$\Leftrightarrow UU^* = I$

$\Leftrightarrow U^{-1} = U^*$

Ex. $U = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

$$U^* U = \begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$

$$\vec{u} \cdot \vec{v} = \vec{u}^* \vec{v} = \vec{a}^* \underbrace{U^* U}_{I} \vec{b} = \vec{a}^* \vec{b} \equiv \vec{a} \cdot \vec{b}$$

$$\equiv \langle \vec{a}, \vec{b} \rangle$$

$\vec{u} = U\vec{a}$

$\vec{v} = U\vec{b}$

Requirement of general inner product

maps $\vec{u}, \vec{v} \in V$ to a complex number $\langle \vec{u}, \vec{v} \rangle$

① $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$

② $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$

③ $c \in \mathbb{C}$, $\langle c\vec{u}, \vec{v} \rangle = \bar{c} \langle \vec{u}, \vec{v} \rangle$

$\langle \vec{u}, c\vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

④ $\langle \vec{u}, \vec{u} \rangle \geq 0$. $\langle \vec{u}, \vec{u} \rangle = 0 \Leftrightarrow \vec{u} = \vec{0}$

Ex. $V = \text{span}\{1, e^{ix}, e^{i2x}\}$

$$e^{ix} = \cos x + i \sin x.$$

Inner prod. $f, g \in V$

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f}(x)\, g(x)\, dx$$

$$\langle f, f \rangle = \int_0^{2\pi} |f(x)|^2 dx \geq 0.$$

Q: $\{1, e^{ix}, e^{i2x}\}$ Orthogonal basis of V?

$$\langle 1, e^{ix} \rangle = \int_0^{2\pi} 1 \cdot e^{ix}\, dx = \int_0^{2\pi} \cos x\, dx + i \int_0^{2\pi} \sin x\, dx$$

$$= 0$$

$$\langle 1, e^{i2x} \rangle = 0$$

$$\langle e^{ix}, e^{i2x} \rangle = \int_0^{2\pi} e^{-ix} \cdot e^{i2x}\, dx = 0.$$

ONB? $\langle 1, 1 \rangle = \int_0^{2\pi} 1\, dx = 2\pi = \langle e^{ix}, e^{ix} \rangle$

$$= \langle e^{i2x}, e^{i2x} \rangle.$$

$$\text{ONB} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} e^{ix}, \frac{1}{\sqrt{2\pi}} e^{i2x} \right\}$$

Fourier series

Real symmetric matrix

$A \in \mathbb{R}^{n \times n}$. $\quad A = A^T$.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$