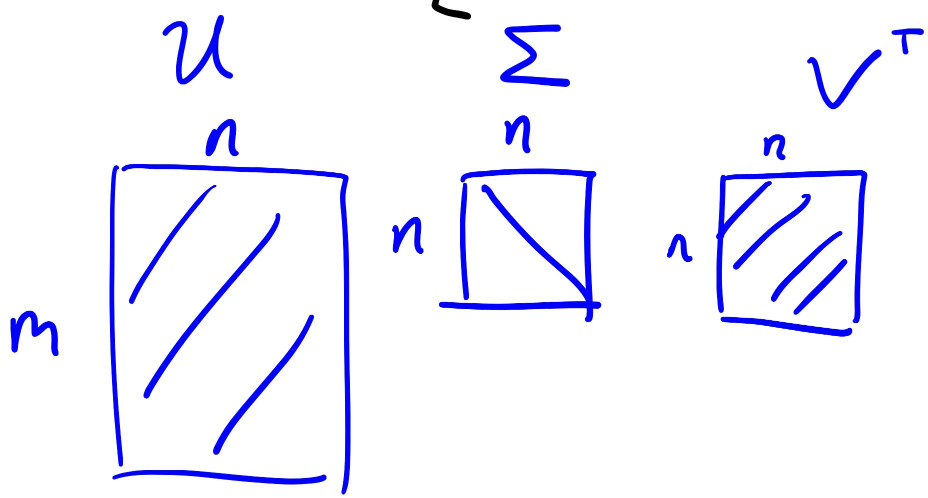
Lec 28 SVD.

How to compute SVD for $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$

$$A = \sum_{k=1}^{n} \vec{u}_k \sigma_k \vec{v}_k^{\,T}$$

$$= [\vec{u}_1 \cdots \vec{u}_n] \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix} \begin{bmatrix} \vec{v}_1^{\,T} \\ \vdots \\ \vec{v}_n^{\,T} \end{bmatrix} = U \Sigma V^T$$



$U^T U = I_n$

$V^T V = I_n$

$$\underbrace{A^T A}_{\text{real sym}} = V\Sigma \underbrace{U^T U}_{I_n} \Sigma V^T = V \Sigma^2 V^T$$

$$\Sigma^2 = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}$$

From the perspective of spec. decomp. thm.

$A^T A$ should be orthogonally diagonalizable.

Computationally,

(1) $A^TA = VDV^T$. $\quad D = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} = \Sigma^2$

(2) Reconstruct $U = AV\Sigma^{-1}$ (assuming $\sigma_i > 0$)

Verify: $U^TU = \Sigma^{-1}V^T \underbrace{A^TA}_{VDV^T} V\Sigma^{-1}$

$$= \Sigma^{-1}D\Sigma^{-1} = I_n.$$

$$\begin{pmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_n^{-1} \end{pmatrix} \begin{pmatrix} \sigma_1^{2} & & \\ & \ddots & \\ & & \sigma_n^{2} \end{pmatrix} \begin{pmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_n^{-1} \end{pmatrix} = I_n$$

<u>Ex</u>. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

① $A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A^T A = V \Sigma^2 V^T$. $\quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \Sigma^2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$$

② $$U = AV\Sigma^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} \end{bmatrix}$$

$$U^T U = I_2 .$$

$$A = U \Sigma V^T .$$

Revisit compression based on SVD.

$$A^T A = V D V^T \qquad D = \begin{bmatrix} \sigma_1^2 & & & & & \\ & \ddots & & & & \\ & & \sigma_K^2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$\sigma_i > 0$

$K < n \qquad \underbrace{\qquad}_{n-K}$

Define $\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_K & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$

$$\Sigma^{+} := \begin{bmatrix} \sigma_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \sigma_K^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$$U = A V \Sigma^{-1} \quad \text{earlier.}$$

$$= \Big[A\vec{v}_1 \sigma_1^{-1} \cdots A\vec{v}_k \sigma_k^{-1} \ \underbrace{\vec{0} \cdots \vec{0}}\Big]$$

Replace by arbitrary columns through Gram-Schmidt process. so that $U^T U = I_n$.

$$A = \sum_{k=1}^{K} \sigma_k \vec{u}_k \vec{v}_k$$

rank-K decomposition.

Address why D can be written as

$$D = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}$$

Recall $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \underbrace{A^T A}_{\text{real symmetric.}} \vec{v}$

$\langle \vec{u}, \vec{v} \rangle$ is an inner product

$\Longleftrightarrow$ All eigenvalues of $A^T A$ are > 0.

Take $\vec{v_i}$ of A^TA.

$$\vec{v_i}^T \underbrace{A^T A}_{\parallel} \vec{v_i} = \langle \vec{v_i}, \vec{v_i} \rangle = \lambda_i \equiv \sigma_i^2$$

$$\left[\begin{array}{l} A^TA\,\vec{v_i} = \lambda_i \vec{v_i} \\ \Rightarrow \vec{v_i}^T A^T A \vec{v_i} = \lambda_i \end{array} \right]$$

$$(A\vec{v_i})^T (A\vec{v_i}) = \|A\vec{v_i}\|^2$$

standard inner product

of $(A\vec{v_i}) \cdot (A\vec{v_i})$ in $\mathbb{R}^n$.

≥ 0.