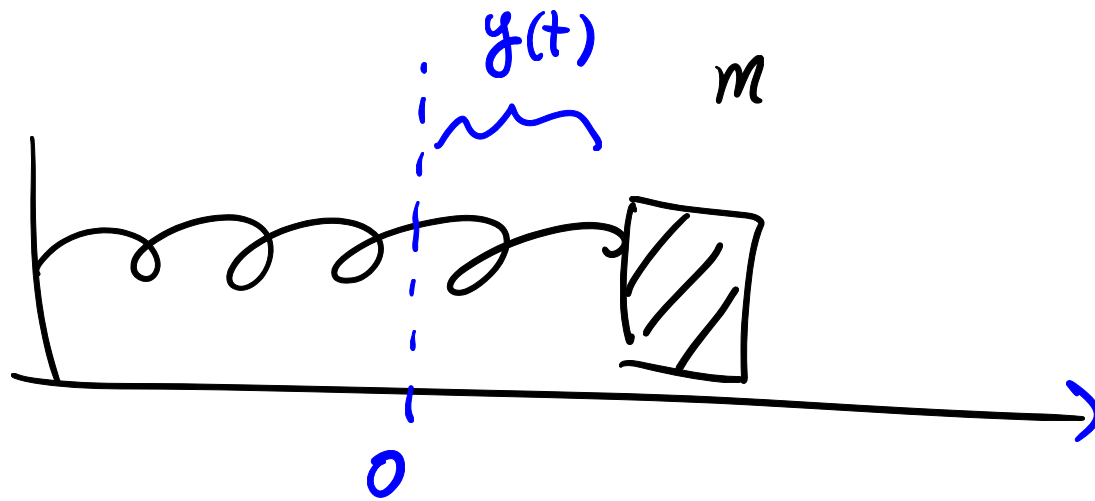


Lec 29. Homogeneous 2nd order ODEs.

Physical system. (spring mass system)



Newton's law

$$m \ddot{y}(t) = \text{FORCE} = F_{\text{spring}} + F_{\text{friction}}$$

$$F_{\text{spring}} = -c y(t) \quad : \text{Hooke's law}$$

$$F_{\text{friction}} = -b \dot{y}(t)$$

$$m \ddot{y}(t) = -b \dot{y}(t) - c y(t)$$

$$\Rightarrow \boxed{m \ddot{y}(t) + b \dot{y}(t) + c y(t) = 0}$$

Hom. 2nd order ODE.

$\neq 0$
in homogeneous.

Initial condition

$$\begin{cases} y(0) = y_0 \\ \dot{y}(0) = y_0' \end{cases}$$

Spring simulator (check course page)

$$m = 1.$$

$$\text{stiffness} : c = 1$$

$$\text{damping} : b \rightarrow \text{vary}$$

$$b^2, \quad 4mc = 4$$

$$\begin{cases} y(0) = -2. \\ \dot{y}(0) = 0 \end{cases}$$

Def. A homogeneous second order
(RHS = 0) (y, \dot{y} , \ddot{y})

ordinary differential equation is an
(y(t))

eq. of the form

$$a \ddot{y}(t) + b \dot{y}(t) + c y(t) = 0$$

If we are given

$$\begin{cases} y(0) = y_0 \\ \dot{y}(0) = y_0' \end{cases}$$

this is called an initial value problem (IVP).

Strategy :

- ① Find general sol. for all initial values
- ② Find specific sol. satisfying initial cond.

Connect to Lin. Alg.

$$V = C^\infty(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{R} \text{ is} \\ \text{infinitely differentiable} \}$$

$$T: V \rightarrow V$$

$$y \mapsto a \ddot{y} + b \dot{y} + c$$

$$T(y) = a \ddot{y} + b \dot{y} + c$$

$$\text{General sol} = \underbrace{\text{Nul}(T)}_{\dim = 2}.$$

$$a y^{(2)} + b y^{(1)} + c y(t) = 0$$

auxiliary eq. $r \in \mathbb{C}$

$$a r^2 + b r + c = 0 \quad \text{Find roots}$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

① $b^2 - 4ac > 0$: 2 distinct real roots

② $b^2 - 4ac = 0$. 1 real root of multiplicity 2

③ $b^2 - 4ac < 0$ 2 distinct complex roots

(conjugate to each other)

Case 1 . Ex. $y'' - 3y' - 4y = 0$

aux. eq. $r^2 - 3r - 4 = 0$

$$r = \frac{3 \pm \sqrt{9 + 16}}{2}$$

$$r_1 = 4, \quad r_2 = -1.$$

Make an educated guess

$$y_1(t) = e^{4t}, \quad y_2(t) = e^{-t}$$

$$y_1' = 4e^{4t} \quad y_1'' = 16e^{4t}$$

$$y_1'' - 3y_1' - 4y_1 = e^{4t} (16 - 12 - 4) = 0$$

$$y_2'' - 3y_2' - 4y_2 = e^{-t} \cdot 0 = 0.$$

$y_1(t), y_2(t)$ lin. indep. in V

i.e., If $c_1 y_1(t) + c_2 y_2(t) = \underbrace{c_1 e^{4t} + c_2 e^{-t}} = 0$

$$\Rightarrow \underbrace{c_1 e^{5t}} = \underbrace{-c_2}$$

not a constant
constant

unless $c_1 = 0 \Rightarrow c_2 = 0$.

$$\text{Nul}(T) = \text{span} \{ \underbrace{e^{4t}, e^{-t}} \}$$

→ basis

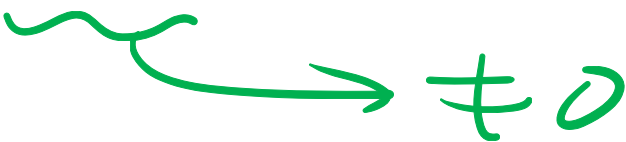
$$= \{ c_1 e^{4t} + c_2 e^{-t} \mid c_1, c_2 \in \mathbb{R} \text{ or } \mathbb{C} \}$$

why?

$$y(t) = e^{rt} \quad \text{for unknown } r \in \mathbb{C}$$

$$\dot{y}(t) = r e^{rt}, \quad \ddot{y}(t) = r^2 e^{rt}$$

$$a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0$$



$$\Leftrightarrow a r^2 + b r + c = 0 \quad \text{aux. eq.}!$$

Ex. $y'' - 2y' + y = 0.$

aux. eq. $r^2 - 2r + 1 = 0. \Rightarrow r = 1$

(w. multiplicity 2)

$$y_1(t) = e^t$$

$$y_2(t) = t e^t$$

$$y_2'(t) = e^t + t e^t = (1+t) e^t$$

$$y_2''(t) = e^t + (1+t)e^t = (2+t)e^t$$

$$y_2'' - 2y_2' + y_2 = e^t [(2+t) - 2(1+t) + t] = 0.$$

$\{e^t, te^t\}$ lin. indep.

$$\text{Nul}(T) = \text{span}\{e^t, te^t\}.$$

