Lec 30. Homogeneous 2nd order ODE. Continued.

Ex. $y'' - y = 0$.

aux. eq. $r^2 - 1 = 0 \Rightarrow r_1 = 1, \ r_2 = -1$.

$y_1(t) = e^{t}, \quad y_2(t) = e^{-t}$

$$T(y) = y'' - y \equiv \left(\frac{d^2}{dt^2} - 1\right) y$$

Fact: $\left(\frac{d}{dt} + a\right)\left(\frac{d}{dt} + b\right) = \frac{d^2}{dt^2} + (a+b)\frac{d}{dt} + ab$

$a, b \in \mathbb{C}$

$$T(y) = \left(\frac{d}{dt} - 1\right)\left(\frac{d}{dt} + 1\right) y = \left(\frac{d}{dt} + 1\right)\left(\frac{d}{dt} - 1\right) y$$

Want : $T(y)=0$.

It is sufficient if

$\left(\frac{d}{dt}+1\right)y=0$ or $\left(\frac{d}{dt}-1\right)y=0$.

$\Updownarrow$ $\Updownarrow$

$y'+y=0$ $y'-y=0$

sol: $y=e^{-t}$ sol: $y=e^{t}$

Ex. $\begin{cases} y'' - y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$ IVP.

General sol: $y(t) = c_1 e^t + c_2 e^{-t}$

Need to determine c_1, c_2. using initial data.

$y(0) = c_1 + c_2 = 1$

$y'(0) = c_1 - c_2 = 0$

$\Rightarrow \begin{cases} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{cases}$

$\Rightarrow y(t) = \frac{1}{2}(e^t + e^{-t})$

Ex. $y'' - 2y' + y = 0$

aux. $r^2 - 2r + 1 = (r-1)^2 = 0.$

$$T(y) = \left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\right)y = \left(\frac{d}{dt} - 1\right)^2 y = 0.$$

It is sufficient to have

$$\left(\frac{d}{dt} - 1\right)y = y' - y = 0 \Rightarrow y(t) = e^t$$

Alternatively

$$\left(\frac{d}{dt} - 1\right)y = e^t$$ ← inhomogeneous 1st order linear ODE.

$$T(y) = \left(\frac{d}{dt} - 1\right)^2 y = \left(\frac{d}{dt} - 1\right)e^t = 0$$

$y(t) = te^t$

General sol $y(t) = c_1 e^t + c_2 te^t$.

Remark: Connect to non-diagonalizable matrix.

Ex. $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\lambda = 3$ (multiplicity 2)

$[A - 3I] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ hom. eq. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ask: which vectors satisfy $(A-3I)^2 \vec{v} = \vec{0}$

Answer: $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

second sol: $(A-3I)\vec{v}_2 = \boxed{\vec{v}_1}$

$$(A-3I)^2\vec{v}_2 = (A-3I)\vec{v}_1 = \vec{0}$$

$$\Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Case 3 : $b^2 - 4ac < 0$

Ex. $y'' - 4y' + 5y = 0$.

aux. $r^2 - 4r + 5 = 0 \Rightarrow r = \dfrac{4 \pm \sqrt{4^2 - 20}}{2} = 2 \pm i$

$$T(y) = \left(\frac{d}{dt} - (2+i)\right)\left(\frac{d}{dt} - (2-i)\right) y = 0.$$

$$y(t) = c_1 e^{(2+i)t} + c_2 e^{(2-i)t}. \quad c_1, c_2 \in \mathbb{C}$$

Ex. IVP. $\begin{cases} y'' - 4y' + 5y = 0 \\ y(0) = 3 \\ y'(0) = -4. \end{cases}$

Only need to plug in initial data.

$$y(0) = c_1 + c_2 = 3$$

$$y'(0) = (2+i)c_1 + (2-i)c_2 = -4$$

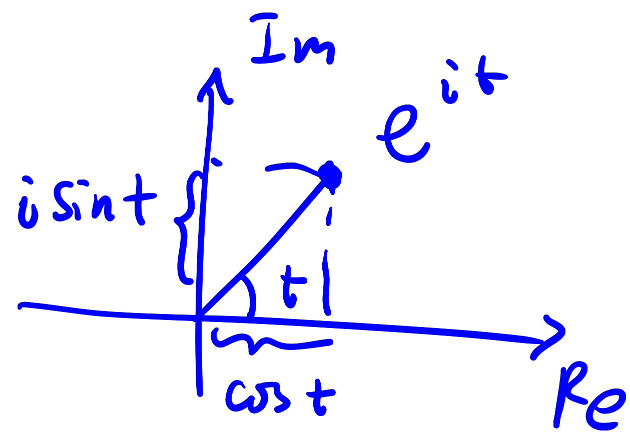
$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + 5i \\ \frac{3}{2} - 5i \end{bmatrix}$$

$$\Rightarrow \quad y(t) = \underbrace{\left(\tfrac{3}{2}+5i\right)e^{(2+i)t}}_{z=a+ib} + \underbrace{\left(\tfrac{3}{2}-5i\right)e^{(2-i)t}}_{\bar{z}=a-ib}$$

$$= 2\,\mathrm{Re}\, z = 2a.$$

Simplify using Euler's formula

$$e^{it} = \cos t + i \sin t$$



$$\Rightarrow y(t) = 3e^{2t}\cos t - 10e^{2t}\sin t \in \mathbb{R}.$$

Q: Is it possible to use real arithmetic only?

YES! Recall general sol.

$$y(t) = c_1 e^{2t} \cdot e^{it} + c_2 e^{2t} e^{-it}$$

$$= e^{2t}\left(c_1 \cos t + i c_1 \sin t + c_2 \cos t - i c_2 \sin t \right)$$

$$= e^{2t}\left[(c_1 + c_2) \cos t + (i c_1 - i c_2) \sin t \right]$$

$$\overset{\text{Rename}}{=} d_1 e^{2t} \cos t + d_2 e^{2t} \sin t$$

We have used.

$$\operatorname{span}_{\mathbb{C}}\{ e^{(2+i)t}, e^{(2-i)t} \} = \operatorname{span}_{\mathbb{C}}\{ e^{2t}\cos t, e^{2t}\sin t \}$$

When $y(0), y'(0) \in \mathbb{R}$

it is simpler to use $\{e^{2t}\cos t, e^{2t}\sin t\}$

as basis, AND let $d_1, d_2 \in \mathbb{R}$

Revisit $\begin{cases} y(0) = 3 \\ y'(0) = -4 \end{cases}$

$$y(0) = d_1 = 3$$

$$y'(0) = d_1 \cdot 2\left(e^{2t}\cos t\right)\Big|_{t=0} + d_2 e^{2t}\left(\cos t\right)\Big|_{t=0}$$

$$= 2d_1 + d_2 = -4$$

$$\Rightarrow \begin{cases} d_1 = 3 \\ d_2 = -10. \end{cases}$$

$$\Rightarrow y(t) = 3e^{2t}\cos t - 10e^{2t}\sin t$$

Real basis is simpler. & is the recommended approach. for computation.

Complex basis is conceptually simpler.

Inhom. 2nd order linear ODE.

The only case relevant for this class:

$$ay'' + by' + cy = t^l e^{rt}, \quad l \in \{0, 1, 2, \dots\}$$

$$r \in \mathbb{C}.$$