

Lec 31. Inhomogeneous 2nd order linear ODEs.

$$y'' + by' + cy = t^l e^{rt}, \quad l \in \mathbb{N}, \quad r \in \mathbb{C}.$$

$$T: y \mapsto y'' + by' + cy$$

Observe

$$(e^{rt})' = r e^{rt}, \quad (e^{rt})'' = r^2 e^{rt}$$

$$(t^a e^{rt})' = a t^{a-1} e^{rt} + r t^a e^{rt}$$

$$\in \text{Span} \{ t^{a-1} e^{rt}, t^a e^{rt} \}.$$

Consider a subspace $W \subset C^\infty(\mathbb{R})$.

$$W := \text{span} \{ e^{rt} \cdot t^m, e^{rt} t^{m-1}, \dots, e^{rt} \}$$

$\dim W = m+1$, intuitively " $W = e^{rt} \cdot \mathbb{P}_m$ "

Our hope: $t^l e^{rt} \in \text{Image}(T) \rightarrow m \geq l$

$$T: W \rightarrow W.$$

$$\begin{aligned} \text{Then } y(t) &= \underline{e^{rt}} p(t), \quad p(t) \in \mathbb{P}_m \\ &= e^{rt} (a_0 + a_1 t + \dots + a_m t^m) \end{aligned}$$

a_0, \dots, a_m are unknown.

Method of undetermined coefficients

$$\begin{aligned} T(y) &= (e^{rt} p(t))'' + b(e^{rt} p(t))' + c e^{rt} p(t) \\ &= (r e^{rt} p(t) + e^{rt} p'(t))' + b(r e^{rt} p(t) + e^{rt} p'(t)) \\ &\quad + c e^{rt} p(t) \\ &= (r^2 e^{rt} p(t) + 2r e^{rt} p'(t) + e^{rt} p''(t)) \\ &\quad + b(r e^{rt} p(t) + e^{rt} p'(t)) + c e^{rt} p(t) \end{aligned}$$

$$\begin{aligned}
 &= e^{rt} \left\{ \underbrace{(r^2 + br + c)}_{\text{red wavy}} p(t) \rightarrow \in \mathbb{P}_m \right. \\
 &\quad + \underbrace{(2r + b)}_{\text{red wavy}} p'(t) \quad \in \mathbb{P}_{m-1} \\
 &\quad \left. + p''(t) \quad \in \mathbb{P}_{m-2} \right\}
 \end{aligned}$$

$$= e^{rt} t^l$$

Case 1: $r^2 + br + c \neq 0$, $m = l$.
 aux. eq. \rightarrow not a root

$$\text{Case 2: } r^2 + br + c = 0$$

$$m = l + 1$$

$$2r + b \neq 0$$

$$\text{Observe } 2r + b = (r^2 + br + c)'$$

$$(r - r_1)(r - r_2) \\ = r^2 + br + c, r_1 \neq r_2$$

$$2r_1 + b = 2r_1 - (r_1 + r_2) \\ = r_1 - r_2 \neq 0$$

$\Rightarrow r$ is a single root of aux. eq.

$$\text{Case 3: } r^2 + br + c = 0$$

$$m = l + 2$$

$$2r + b = 0$$

r is a double root of aux. eq.

$$\text{double root: } (r - r_1)^2 = r^2 + br + c. \quad b = -2r_1 \Rightarrow 2r_1 + b = 0$$

Ex. $y'' + 2y' + y = t^2$.

Step 1: $t^2 = t^2 \cdot e^{0t}$, $r=0$.

Step 2: plug in $r=0$ into

$$r^2 + 2r + 1 = 1 \neq 0.$$

$\Rightarrow r=0$ is not a root

should choose $m=2$,

$$W = \text{span} \left\{ \underbrace{t^2, t, 1}_B \right\}.$$

Step 3. Solve this as a lin. sys. problem.

$$[t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T: W \rightarrow W$$

$$[T]_{\mathcal{B}} = \left[[T(\vec{b}_1)]_{\mathcal{B}} \quad [T(\vec{b}_2)]_{\mathcal{B}} \quad [T(\vec{b}_3)]_{\mathcal{B}} \right]$$

$$T(\vec{b}_1) = T(t^2) = 2 + 4t + t^2$$

$$[T(\vec{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{aligned} T(t^2) &= (t^2)'' + 2(t^2)' + t^2 \\ &= 2 + 4t + t^2 \end{aligned}$$

Similarly

$$[T(\vec{b}_2)]_B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad [T(\vec{b}_3)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \end{array} \right] \rightarrow [\vec{y}]_B = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

A Solution: $y_p(t) = t^2 - 4t + 6$

All solutions

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

$$+ y_p(t)$$

fixed by
initial condition.

Recall lin sys.

$$A\vec{x} = \vec{b}, \quad \textcircled{1} \text{ Find } \vec{x}_p, \text{ s.t. } A\vec{x}_p = \vec{b}$$

$$\textcircled{2} \text{ sol set} = \{ \vec{x}_p + \vec{w} \mid A\vec{w} = \vec{0} \}.$$

Ex . $y'' + 3y' + 2y = e^{-2t}$

Step 1: $e^{-2t} = t^l e^{rt}$ with $l=0, r=-2$

Step 2 : $r = -2$ is a single root
of aux eq. $m = l + 1 = 1$.

$$W = \text{span} \left\{ \underbrace{te^{-2t}, e^{-2t}}_B \right\}.$$

Step 3:

$$T(\vec{b}_1) = (te^{-2t})'' + 3(te^{-2t})' + 2(te^{-2t})$$

$$= -e^{-2t}$$

$$T(\vec{b}_2) = 0$$

$$[T(\vec{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad [T(\vec{b}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[e^{-2t}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solve $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 0 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{c} \vec{y} \\ \vec{y}_p \end{array} \right]_{\mathbb{B}} = \left[\begin{array}{c} -1 \\ 0 \end{array} \right]$

$$\vec{y}_p(t) = -t e^{-2t}$$

Sol set $\vec{y}(t) = \underbrace{C_1 e^{-2t} + C_2 e^{-t}}_{\text{hom. part}} - t e^{-2t}$

Ex. $y'' - 2y' + y = e^t$

Step 1. $e^t = t^l e^{rt}$, $l=0, r=1$.

Step 2. $r=1$ is a double root
of aux. eq.

$$m = l + 2 = 2$$

$$W = \text{span} \{ \underbrace{t^2 e^t, t e^t, e^t}_{B} \}.$$

Step 3.

$$T(\vec{b}_1) = 2e^t \quad . \quad [T(\vec{b}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$T(\vec{b}_2) = T(\vec{b}_3) = 0 \quad .$$

$$[e^t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Solve } \begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ \underline{2} & 0 & 0 & | & 1 \end{bmatrix} \rightarrow [\vec{y}_p]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$y_p(t) = \frac{1}{2} t^2 e^t .$$

hom. part

$$\text{All sol : } y(t) = c_1 e^t + c_2 t e^t + \frac{1}{2} t^2 e^t$$

