Lec 32. Inhomogeneous 2nd order ODE.

Ex. $y'' + 2y' + y = t^2$.

$t^2 e^{0t}$, $r = 0$, $r^2 + 2r + 1 = 0^2 + 2\cdot 0 + 1 = 1 \neq 0$.

ansatz: $y(t) = a_2 t^2 + a_1 t + a_0$.

Plug into eq.

$y''(t) = 2a_2$, $y'(t) = 2a_2 t + a_1$

$$\Rightarrow \quad 2a_2 + 2(2a_2 t + a_1) + (a_2 t^2 + a_1 t + a_0) = t^2$$

$$\Rightarrow \quad a_2 t^2 + (a_1 + 4a_2)t + (2a_2 + 2a_1 + a_0) = t^2$$

Match coefficients.

$$\begin{cases} a_2 = 1 \\ 4a_2 + a_1 = 0 \\ 2a_2 + 2a_1 + a_0 = 0 \end{cases} \quad \longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right]$$

$a_2 = 1 \qquad a_1 = -4a_2 = -4, \quad a_0 = -2a_2 - 2a_1 = -2 + 8 = 6$

$$\Rightarrow \quad y(t) = t^2 - 4t + 6$$

Ex. $y'' + 3y' + 2y = 20e^{3t}$

$t^l e^{rt} = e^{3t}$: $l = 0$, $r = 3$.

$r^2 + 3r + 2 = 3^2 + 3 \cdot 3 + 2 \neq 0$.

ansatz: $y(t) = a_0 e^{3t}$

$T: W \to W$

$W = \operatorname{span}\{e^{3t}\}$

$y'' + 3y' + 2y = a_0 e^{3t}(9 + 3 \cdot 3 + 2) = 20 a_0 e^{3t} = 20 e^{3t}$

$\Rightarrow a_0 = 1$. $y(t) = e^{3t}$.

$\underline{Ex}$. $y'' + 3y' + 2y = e^{-2t}$

$t^l e^{rt} = e^{-2t}$, $l = 0$, $r = -2$.

$r^2 + 3r + 2 = (r+1)(r+2)$, 2 is a single root

ansatz: $y(t) = a_1 t e^{-2t} \left(+ a_0 e^{-2t} \right)$ → hom. eq.

$y' = a_1 e^{-2t} - 2a_1 t e^{-2t} + a_0 (-2) e^{-2t} = e^{-2t} \left(-2a_1 t + a_1 - 2a_0 \right)$

$y'' = -2(a_1 - 2a_0) e^{-2t} - 2a_1 \left(e^{-2t} + t(-2) e^{-2t} \right)$

$= e^{-2t} \left(4a_1 t - 4a_1 + 4a_0 \right)$

$$y'' + 3y' + 2y = e^{-2t}\left(4\cancel{a_1} t - 4a_1 + 4\cancel{a_0} - 6\cancel{a_1} t + 3a_1 - 6\cancel{a_0} + 2\cancel{a_1} t + 2\cancel{a_0}\right)$$

$$= e^{-2t}(-a_1) = e^{-2t} \Rightarrow a_1 = -1$$

$$a_0 \in \mathbb{R}$$

$$\Rightarrow y_p(t) = -t e^{-2t}.$$

General sol. $y(t) = -te^{-2t} + c_1 e^{-2t} + c_2 e^{-t}.$