Lec 33. Inhomogeneous 2nd order ODEs.

Ex. $y'' = \cos t$. $\longrightarrow$ not $t^{l} e^{rt}$?

$$\begin{cases} e^{it} = \cos t + i\sin t \\ e^{-it} = \cos t - i \sin t \end{cases} \longrightarrow \begin{aligned} \cos t &= \frac{1}{2}\left(e^{it} + e^{-it}\right) \\ \sin t &= \frac{1}{2i}\left(e^{it} - e^{-it}\right) \end{aligned} \quad \begin{aligned} r &= i \\ r &= -i \end{aligned}$$

Answer for a particular sol:

$$y_p(t) = -\cos t$$

general sol: $y(t) = -\cos t + \underbrace{c_1 + c_2 t}_{\text{sol to } y''=0}$.

Superposition principle (linear combination of RHS)

Ex. $y'' + 3y' + 2y = \underbrace{t^3} + \underbrace{e^t}$ $\quad (*)$

Consider
$$\begin{cases} y_1'' + 3y_1' + 2y_1 = t^3 & \rightarrow \quad y_1(t) \\ y_2'' + 3y_2' + 2y_2 = e^t & \rightarrow \quad y_2(t) \end{cases}$$

$y(t) = y_1(t) + y_2(t)$ will satisfy $(*)$

$$y'' + 3y' + 2y = (y_1'' + 3y_1' + 2y_1) + (y_2'' + 3y_2' + 2y_2) = t^3 + e^t$$

Now solve $y'' = \cos t = \frac{1}{2}(e^{it} + e^{-it})$

in a deliberately complex way.

$y_1'' = \frac{1}{2} e^{it}$ $\qquad r = i \qquad r^2 = -1 \neq 0$

$y_1(t) = a_0 e^{it} \in \operatorname{span}\{e^{it}\}$.

$a_0 (e^{it})'' = a_0 (+i)^2 e^{it} = -a_0 \cancel{e^{it}} = \frac{1}{2} \cancel{e^{it}}$

$\Rightarrow a_0 = -\frac{1}{2}$

$y_2'' = \frac{1}{2} e^{-it}$, $\quad y_2(t) = b_0 e^{-it} \in \operatorname{span}\{e^{-it}\}$

$$\Rightarrow \quad b_0 = -\frac{1}{2}$$

$$\Rightarrow \quad y_p(t) = y_1(t) + y_2(t) = -\frac{1}{2}\left(e^{it} + e^{-it}\right) = -\cos t.$$

Ex. $\quad y'' = \underline{t \cos t} = \frac{1}{2} t e^{it} + \frac{1}{2} t e^{-it}$

$$y_1'' = \frac{1}{2} t e^{it} \qquad . \ r = i. \qquad r^2 = -1 \neq 0$$

$$y_1(t) = a_0 e^{it} + a_1 t e^{it} \in \text{span}\{e^{it}, \ t e^{it}\}.$$

$$y_1'(t) = i\, a_0 e^{it} + a_1 t (i) e^{it} + a_1 e^{it}$$

$$y_1''(t) = -\, a_0 e^{it} + a_1 i e^{it} + a_1 t (-1) e^{it} + i\, a_1 e^{it}$$

$$= e^{it}\left(-a_1 t + 2i a_1 - a_0\right) = \frac{1}{2} t\, e^{it}$$

$$\Rightarrow \begin{cases} -a_1 = \frac{1}{2} \\ 2i a_1 - a_0 = 0 \end{cases} \Rightarrow \begin{cases} a_0 = -i \\ a_1 = -\frac{1}{2} \end{cases}$$

$$y_2''(t) = \frac{1}{2} t e^{-it}$$

$$y_2(t) = b_0 e^{-it} + b_1 t e^{-it} \in \operatorname{span}\{e^{-it}, t e^{-it}\}.$$

$$y_2''(t) = e^{-it}\left(-b_1 t - 2i b_1 - b_0\right) = \frac{1}{2} t e^{-it}$$

$$\Rightarrow \begin{cases} -b_1 = \frac{1}{2} \\ -2i b_1 - b_0 = 0 \end{cases} \Rightarrow \begin{cases} b_0 = i \\ b_1 = -\frac{1}{2} \end{cases}$$

$$y(t) = y_1(t) + y_2(t)$$

$$= -ie^{it} - \frac{1}{2}te^{it} + ie^{-it} - \frac{1}{2}te^{-it}$$

$$= -t\cos t + 2\sin t$$

$$\left[-i\left(e^{it} - e^{-it}\right) = -i\left(2i\sin t\right)\right].$$

Verify:

$$y'(t) = -\cos t - t(-\sin t) + 2\cos t$$

$$= t\sin t + \cos t$$

$$y''(t) = \sin t + t\cos t - \sin t$$

$$= t\cos t$$

Similar to hom. case. Complex arithmetics is conceptually simpler.

but there are other ways that are computationally simpler.

In previous example.

$$y(t) = a_0 e^{it} + a_1 t e^{it} + b_0 e^{-it} + b_1 t e^{-it}$$

$$a_0, a_1, b_0, b_1 \in \mathbb{C}.$$

Use Euler's formula

$$= (a_0+b_0)\cos t + (ia_0 - ib_0)\sin t$$

$$+ (a_1+b_1)\,t\cos t + (ia_1 - ib_1)\,t\sin t$$

$$= C_0 \cos t + S_0 \sin t$$

$$+ C_1\, t\cos t + S_1\, t\sin t$$

$C_0, S_0, C_1, S_1 \in \mathbb{R}$.

$\in \operatorname{span}\{\cos t, \sin t, t\cos t, t\sin t\}.$

Ex. $y'' + 2y' - 3y = 10e^{-3t}$

$r = -3$, $\quad r^2 + 2r - 3 = (r+3)(r-1)$

$\uparrow$

single root.

$$y_p(t) = \underbrace{\cancel{a_0 e^{-3t}}}_{\text{sol hom. eq.}} + a_1 t e^{-3t}$$

$\underline{Ex}$. $y'' + 2y' - 3y = 2t^2 e^t$

$r = 1$ single root.

$$y_p(t) = \cancel{a_0 e^t} + a_1 t e^t + a_2 t^2 e^t + a_3 t^3 e^t$$

$\underline{Ex}$. $y'' + 2y' - 3y = e^t \cos t$

$$= e^t \tfrac{1}{2}(e^{it} + e^{-it})$$

$$= \tfrac{1}{2} e^{(1+i)t} + \tfrac{1}{2} e^{(1-i)t}$$

$1 \pm i$ is not a root.

$$y_p(t) = a_0 e^{(1+i)t} + b_0 e^{(1-i)t}, \quad a_0, b_0 \in \mathbb{C}$$

$$= c_0 e^t \cos t + s_0 e^t \sin t, \quad c_0, s_0 \in \mathbb{R}.$$