Lec 34. First order linear diff. eq. system

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad A\vec{x} = \vec{b}$$

$$\frac{d}{dt}\vec{y}(t) = A\,\vec{y}(t) + \vec{f}(t) \qquad \vec{y}(t),\ \vec{f}(t) \in \mathbb{R}^n$$

$$A \in \mathbb{R}^{n\times n}.$$

i.e.,

$$\begin{cases} \frac{d}{dt} y_1(t) = a_{11}y_1(t) + \cdots + a_{1n}y_n(t) + f_1(t) \\ \vdots \\ \frac{d}{dt} y_n(t) = a_{n1}y_1(t) + \cdots + a_{nn}y_n(t) + f_n(t) \end{cases} \qquad (*)$$

normal form

Ex. $y'' + by' + cy = 0$

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \end{cases} \longrightarrow \begin{cases} y_1'(t) = y'(t) = \underline{\underline{y_2(t)}} \\ y_2'(t) = y''(t) = -by'(t) - cy(t) \end{cases}$$

$$= -b\,y_2(t) - c\,y_1(t)$$

$$\Rightarrow \vec{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \qquad \vec{y}\,'(t) = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \vec{y}(t)$$

Ex. $y'' + by' + cy = t^l e^{rt}$

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \end{cases}$$

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y''(t) = -b\,y_2(t) - c\,y_1(t) + \boxed{t^l e^{rt}} \end{cases}$$

$$\vec{f}(t) = \begin{bmatrix} 0 \\ t^l e^{rt} \end{bmatrix}$$

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}\vec{y}(t) + \vec{f}(t).$$

Ex. $y''' - y' + 2y = 0$.

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \\ y_3(t) = y''(t) \end{cases} \longrightarrow \begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y_3(t) \\ y_3'(t) = y' - 2y = y_2(t) - 2y_1(t) \end{cases}$$

$$\Rightarrow \quad \frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \vec{y}(t)$$

Ex. $y'''' - y'' = e^{rt}$

$$\begin{cases} y_1(t) = y(t) \\ y_2(t) = y'(t) \\ y_3(t) = y''(t) \\ y_4(t) = y'''(t) \end{cases} \longrightarrow \begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y_3(t) \\ y_3'(t) = y_4(t) \\ y_4'(t) = y'' + e^{rt} \end{cases}$$

$$= y_3(t) + e^{rt}$$

$$\frac{d}{dt}\vec{y}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \vec{y}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{rt} \end{bmatrix}$$

Initial value problem.

$$\begin{cases} \frac{d}{dt}\vec{y}(t) = A\,\vec{y}(t) + \vec{f}(t) \\ \vec{y}(t_0) = Y_0 \in \mathbb{R}^n. \end{cases}$$

Only case relevant for this class:

A is diagonalizable. $\vec{f}(t) = 0$

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t).$$

Recall $A \in \mathbb{R}^{n\times n}$ is diagonalizable.

$\vec{v}_1, \cdots, \vec{v}_n \in \mathbb{R}^n$. form a basis of $\mathbb{R}^n$.

$$A\vec{v}_i = \lambda_i \vec{v}_i, \quad i = 1, \cdots, n.$$

Observe (λ_i, v_i).

$$\vec{y}_i(t) = e^{\lambda_i t}\vec{v}_i$$

$$\frac{d}{dt}\vec{y}_i(t) = \lambda_i e^{\lambda_i t}\vec{v}_i$$

$$A\vec{y}_i(t) = e^{\lambda_i t} A\vec{v}_i = \lambda_i e^{\lambda_i t}\vec{v}_i$$

So. $\vec{y}_i(t) = e^{\lambda_i t}\vec{v}_i$ is a sol.

$\mathrm{span}\{e^{\lambda_1 t}\vec{v_1}, \cdots, e^{\lambda_n t}\vec{v_n}\}$ is the sol set.

Fact. this set is lin. indep.

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

diagonalize A.

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \Rightarrow \lambda = \pm 1.$$

$\lambda_1 = 1$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

sol set span $\left\{ e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.