"Lec" 35. Review 1.

Part 1. Lin alg.

$A\vec{x} = \vec{b}$, $\quad A \in \mathbb{R}^{m \times n}$, $\vec{x} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^m$

$A = [\vec{a}_1, \cdots, \vec{a}_n] \quad \vec{a}_j \in \mathbb{R}^m$

$$\Leftrightarrow \quad \sum_{j=1}^{n} x_j \vec{a}_j = \vec{b}$$

Row reduction.

Sol set?

① no sol. sol set ϕ

② unique sol.

③ more than 1 sol. $\longrightarrow$ infinite # of sols.

parametric form.

$\downarrow$

span.

$A\vec{x} = \vec{0}$ homogeneous eq.

$A\vec{x} = \vec{b}$ inhom. eq.

Sol set $\{ \vec{x}_p + \vec{w} \mid A\vec{w} = \vec{0} \}$.

$A\vec{x}_p = \vec{b}$

Linearity

$$A(\vec{x}_p + \vec{w}) = A\vec{x}_p + A\vec{w} = \vec{b} + \vec{0} = \vec{b}$$

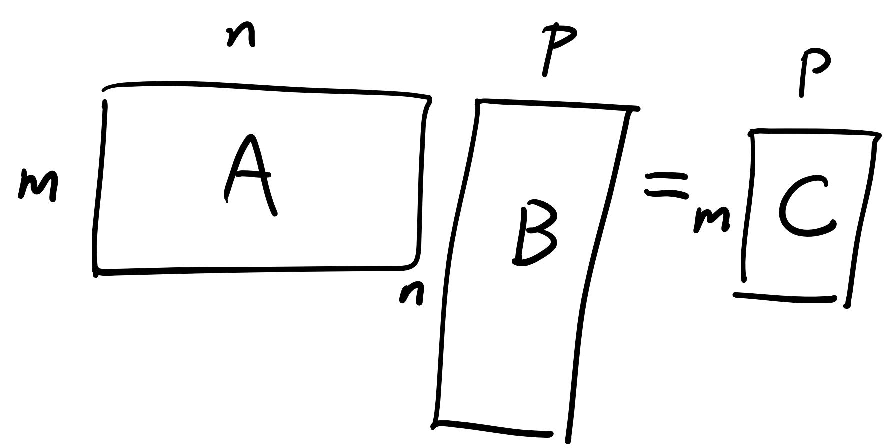
$$\left.\begin{aligned} A\vec{x_1} &= \vec{b} \\ A\vec{x_2} &= \vec{b}\end{aligned}\right\} \Rightarrow A(\vec{x_1}-\vec{x_2}) = \vec{0}$$

$$\vec{x_1}-\vec{x_2} = \vec{w} \quad \text{s.t. } A\vec{w}=\vec{0}$$

<u>Matrix</u>. $A \in \mathbb{R}^{m\times n}$. square $A \in \mathbb{R}^{n\times n}$

$$AB = C$$

$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$



Linear transformation.

matrix multiplication. $\Leftrightarrow$ composition of lin. trans.

$$T_1 : \mathbb{R}^p \to \mathbb{R}^n. \qquad \underset{\mathbb{R}^p}{\underset{\cap}{\vec{x}}} \mapsto \underset{\mathbb{R}^n}{\underset{\cap}{B\vec{x}}}$$

$$T_2 : \mathbb{R}^n \to \mathbb{R}^m. \qquad \underset{\mathbb{R}^n}{\underset{\cap}{\vec{y}}} \mapsto \underset{\mathbb{R}^m}{\underset{\cap}{A\vec{y}}}$$

$$\boxed{C = AB}$$

$$T_3 = T_2 \circ T_1 : \mathbb{R}^p \to \mathbb{R}^m \qquad \underset{\mathbb{R}^p}{\underset{\cap}{\vec{x}}} \mapsto \underset{\mathbb{R}^m}{\underset{\cap}{C\vec{x}}}$$

Invertibility.

$A \in \mathbb{R}^{n \times n}$

$A A^{-1} = I_n$

how to find?

Solve!

$I_n = [\vec{e}_1 \cdots \vec{e}_n]$

$A \vec{x}_j = \vec{e}_j \ , \ j = 1, \cdots, n$

$A^{-1} = [\vec{x}_1 \cdots \vec{x}_n]$

$T : \vec{x} \mapsto A\vec{x}$

$T^{-1} : \vec{y} \mapsto A^{-1}\vec{y}$.

$T \circ T^{-1}(\vec{y}) = \vec{y}$

$T^{-1} \circ T(\vec{x}) = \vec{x}$

How to determine a matrix is invertible?

$\det(A) \neq 0$

$\Updownarrow$

A is invertible.

$\Updownarrow$

$RREF = I_n$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

cofactor expansion.

$A = [\vec{a}_1, \dots, \vec{a}_n] \qquad \vec{a}_j \in \mathbb{R}^m.$

set $\{\vec{a}_1, \dots, \vec{a}_n\}$ lin. indep ?

Ex. $\vec{a}_1, \vec{a}_2, \vec{a}_3 = \vec{a}_1 + 2\vec{a}_2$

span $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} =$ span $\{\vec{a}_1, \vec{a}_2\}$

$x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$ has only trivial sol.

$A\vec{x} = \vec{0}$. hom. eq.

$A \in \mathbb{R}^{n \times n}$.

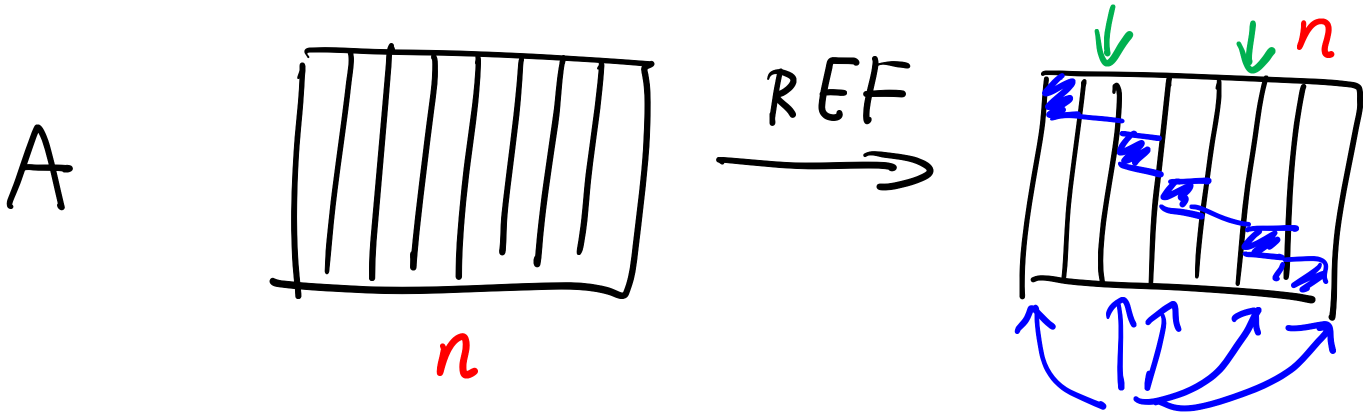
$\{\vec{a}_1, \dots, \vec{a}_n\}$ lin. indep. one-to-one

$\Rightarrow$ A is invertible : one-to-one & onto.

$\text{span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathbb{R}^n$. onto

Rank thm.

no pivot. # free var = dim Nul(A)



pivots = rank(A) = dim Col(A)

$$n = \underset{\text{no pivot}}{\dim Nul(A)} + \underset{\text{pivot}}{\dim Col(A)}$$

For $A \in \mathbb{R}^{n \times n}$.

one-to-one $\Rightarrow \dim Nul(A) = 0 \Rightarrow \mathrm{rank}(A) = n$

$\Rightarrow$ onto

onto $\Rightarrow \mathrm{rank}(A) = n \Rightarrow \dim Nul(A) = 0 \Rightarrow$ one-to-one