Lec 4.

$$\left[\begin{array}{ccc|c} 2 & -5 & 8 & 5 \\ -2 & -4 & 1 & -5 \\ 4 & -1 & 7 & 10 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 2 & -5 & 8 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2x_1 - 5x_2 + 8x_3 = 5$$

x_3 is free var.

$$x_2 - x_3 = 0.$$

$$\begin{cases} x_1 = \frac{5}{2} - \frac{3}{2}x_3 \\ x_2 = x_3 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ 0 \end{bmatrix} + \underbrace{x_3 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \end{bmatrix}}$$

special sol $x_3=0$.

$\longrightarrow$ sols of hom. lin. sys.

$$[A \mid b] \iff x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n = \vec{b}$$

admits 2 sols. $\vec{x} = \vec{u}$, $\vec{x} = \vec{v}$

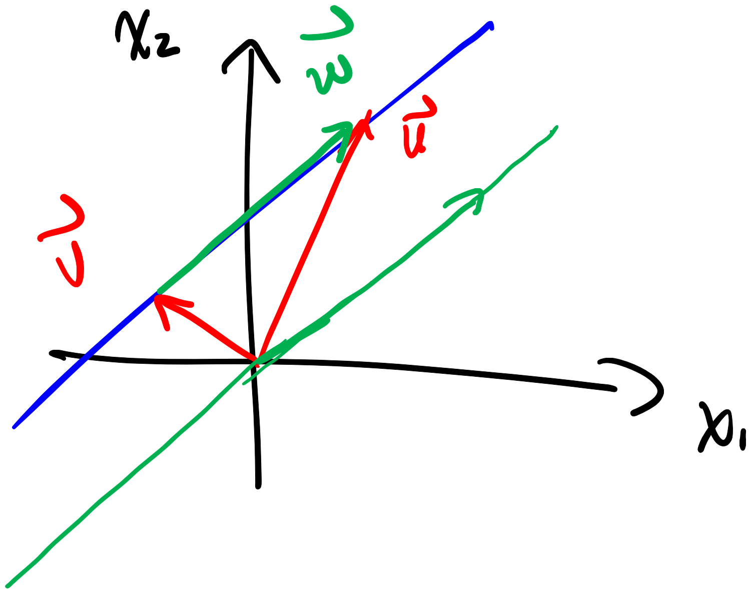
$$\vec{w} = \vec{u} - \vec{v}$$

$$\begin{cases} u_1 \vec{a}_1 + \cdots + u_n \vec{a}_n = \vec{b} \\ v_1 \vec{a}_1 + \cdots + v_n \vec{a}_n = \vec{b} \end{cases}$$

$$(u_1 - v_1)\vec{a}_1 + \cdots + (u_n - v_n)\vec{a}_n = \vec{0}$$

$\Rightarrow$ $\vec{w} = \vec{u} - \vec{v}$ is sol. to. hom. lin. sys.

i.e., $A\vec{w} = \vec{0}$.



geometric perspective

Def The span of $\vec{v}_1, \cdots, \vec{v}_k$ is the set of all vectors written as lin. combination of $\vec{v}_1, \cdots, \vec{v}_k$

$$\operatorname{span}\{\vec{v}_1, \cdots, \vec{v}_k\} := \{a_1\vec{v}_1 + \cdots + a_k\vec{v}_k \mid a_1, \cdots, a_k \in \mathbb{R}\}$$

Ex. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix} \right\}$?

$$\Leftrightarrow \left[\begin{array}{cc|c} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 0 & 6 & 3 \end{array}\right] \text{ has sol?}$$

$$\xrightarrow[\text{exer}]{\text{REF}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \qquad \text{Yes.}$$

Linear dependence $\Longleftrightarrow$ redundancy

<u>Def</u> A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly independent if

$$x_1 \vec{v}_1 + \cdots + x_k \vec{v}_k = \vec{0}$$

has only trivial sol.

Otherwise linearly dependent.

Ex. $\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix}$

lin. dep?

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ -2 & -6 & 2 & 0 \\ 4 & 7 & -4 & 0 \end{array}\right] \xrightarrow[\text{exer}]{\text{REF}} \left[\begin{array}{ccc|c} \boxed{2} & 4 & -2 & 0 \\ 0 & \boxed{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ x_3 is free var → inf. sol.

→ lin. dep.

Sol set $\left\{ x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$.

$$x_3 \vec{v_1} + 0 \cdot \vec{v_2} + x_3 \vec{v_3} = \vec{0}$$

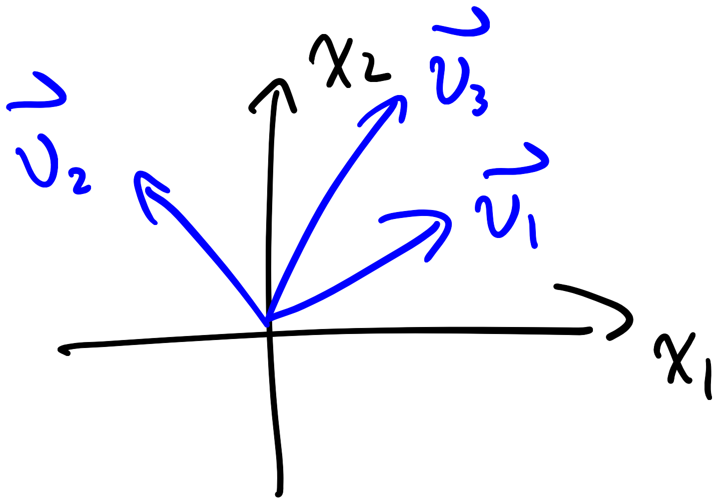
Take $x_3 = 1$,

$$\vec{v_1} = \underbrace{0 \cdot \vec{v_2} - 1 \cdot \vec{v_3}}_{\text{lin. comb. of } \vec{v_2},\, \vec{v_3}}$$

$$\operatorname{span}\{\vec{v_1}, \vec{v_2}, \vec{v_3}\} = \operatorname{span}\{\vec{v_2}, \vec{v_3}\}$$

$$= \operatorname{span}\{\vec{v_1}, \vec{v_2}\} \neq \operatorname{span}\{\vec{v_1}, \vec{v_3}\}$$

Geometric



Thm. A set of vectors $\{\vec{v_1}, \cdots, \vec{v_k}\}$ is lin. dep.

$\Leftrightarrow$ at least one of the vectors is a lin. comb. of the rest of vectors.

Proof: ① $\Leftarrow$ there exists $\vec{v_p}$, $1 \le p \le k$.

$$\vec{v_p} = \sum_{\substack{i=1 \\ i \neq p}}^{k} c_i \vec{v_i}$$

$$\Rightarrow \quad c_1\vec{v_1} + c_2\vec{v_2} + \cdots + (-1)\cdot\vec{v_p} + c_{p+1}\vec{v_{p+1}} + \cdots + c_k\vec{v_k} = \vec{0}$$

has nontrivial sol. $\Rightarrow$ lin. dep.

② $\Rightarrow$ there exists $\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \neq \vec{0}$ s.t.

$$\sum_{i=1}^{k} c_i \vec{v_i} = \vec{0}$$

say $c_p \neq 0$. $(1 \le p \le k)$

$$\vec{v_p} = \sum_{\substack{i=1 \\ i \neq p}}^{k} \left(-\frac{c_i}{c_p}\right)\vec{v_i} \qquad \square$$