# Lec 6

$$A \in \mathbb{R}^{m \times n} \qquad A = \begin{array}{cc} & n \\ m & \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \end{array}$$

**Def** $A, B \in \mathbb{R}^{m \times n}$

(i) $A = B$, $A_{ij} = B_{ij}$, $\forall\ 1 \le i \le m,\ 1 \le j \le n$

(ii) $C = A + B$, $C_{ij} = A_{ij} + B_{ij}$ " "

(iii) $x \in \mathbb{R}$, $C = xA$, $C_{ij} = x A_{ij}$, " "

(iv) $C = x_1 A + x_2 B \in \mathbb{R}^{m \times n}$

$$C_{ij} = x_1 A_{ij} + x_2 B_{ij}.$$

---

Matrix multiplication

Proposal: $A, B \in \mathbb{R}^{m \times n}$

$C = AB$ is defined as $C_{ij} = A_{ij} B_{ij}$

NOT what we need.

Linear transformation $\Longleftrightarrow$ matrix trans.

$T : \mathbb{R}^n \to \mathbb{R}^m$ $\qquad$ $A \in \mathbb{R}^{m \times n}$

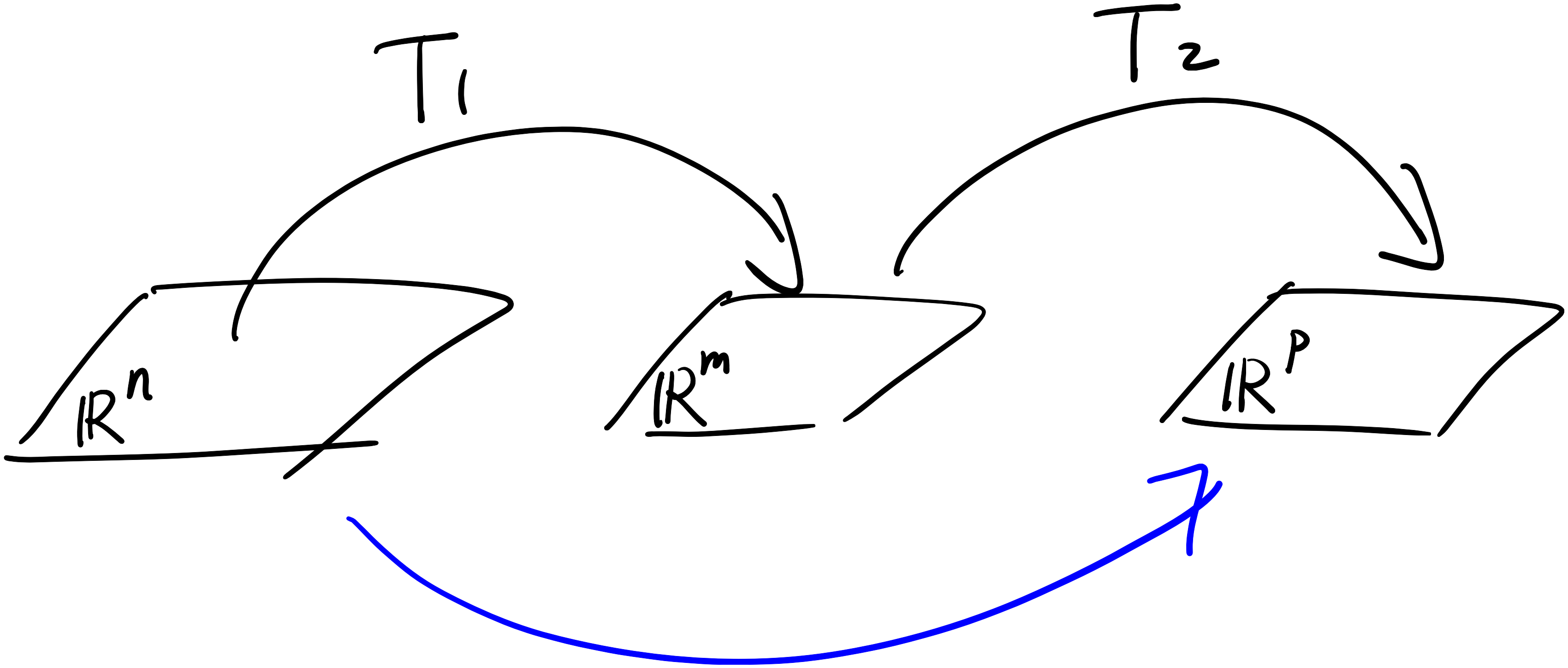
$$T(\vec{x}) = A\vec{x}$$

Composition of trans.

$T_1 : \mathbb{R}^n \to \mathbb{R}^m$ , $\qquad$ $T_2 : \mathbb{R}^m \to \mathbb{R}^p$

$\downarrow$ std. matrix $\qquad$ $\downarrow$ std. matrix

$B \in \mathbb{R}^{m \times n}$ $\qquad$ $A \in \mathbb{R}^{p \times m}$




$$\underbrace{T_3} = T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p$$

Linear.

$\downarrow$ std. matrix?

$C \in \mathbb{R}^{p \times n}$

Would like to find $C_{ij}$

$$C = [\vec{c}_1 \ \cdots \ \vec{c}_n] \qquad \vec{c}_j \in \mathbb{R}^p$$

$$\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{th}$$

$$C\vec{e}_j = \vec{c}_j$$

$$\vec{c}_j = (T_2 \circ T_1)(\vec{e}_j) \equiv T_2\left(T_1(\vec{e}_j)\right)$$

$$= T_2\Big(\underbrace{B\vec{e}_j}_{j\text{th col of } B}\Big)$$

$$= A\,(B\vec{e}_j)$$

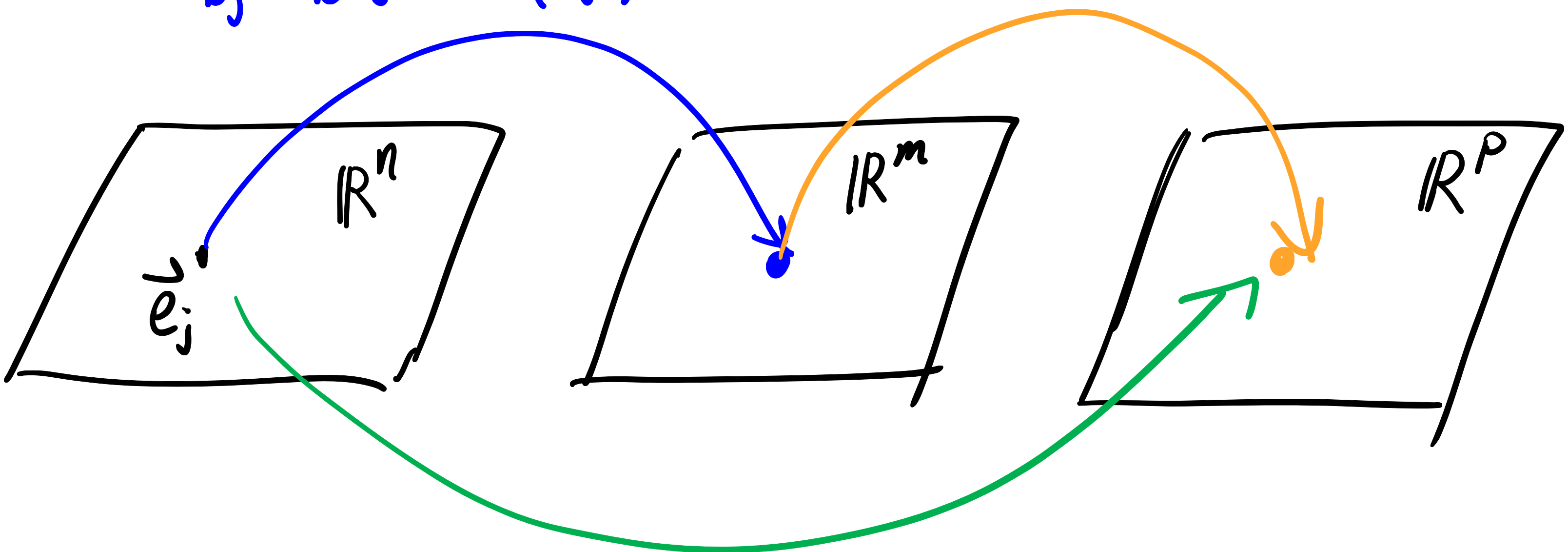
Vector multiplication $\underline{\underline{\quad}}$ $\vec{a}_1 B_{1j} + \vec{a}_2 B_{2j} + \cdots + \vec{a}_m B_{mj}$

$$C_{ij} = (\vec{C}_j)_i = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{im}B_{mj}$$

$$\boxed{C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}}$$

$\vec{b}_j = B\vec{e}_j = T_1(\vec{e}_j)$

$T_2(\vec{b}_j) = A\vec{b}_j = A(B\vec{e}_j)$

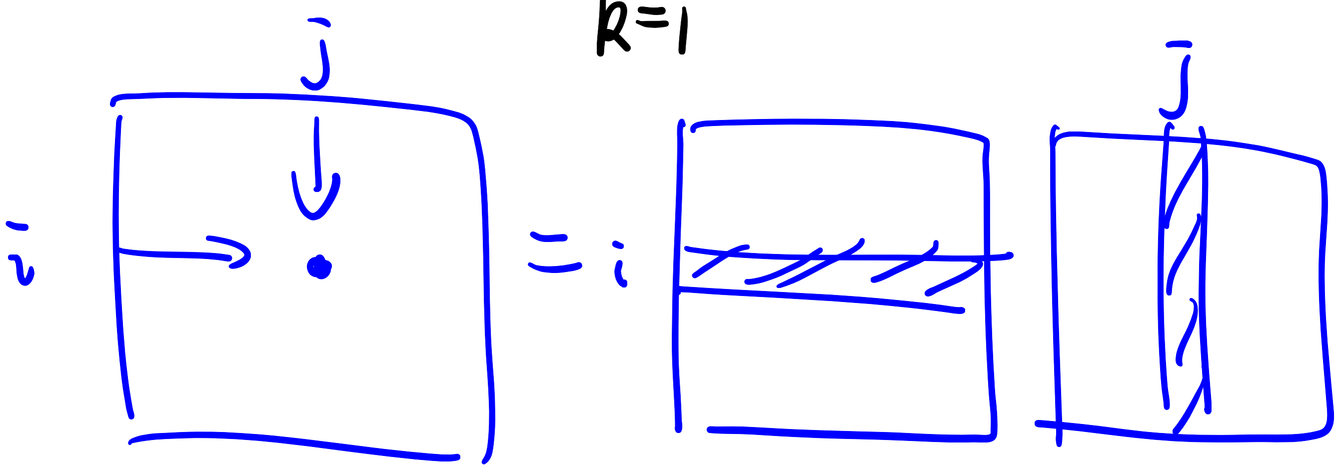



$$\vec{c}_j = (T_2 \circ T_1)(\vec{e}_j) = \underbrace{(AB)}_{\text{matrix multiplication}}\vec{e}_j$$

Vector multiplication is a special case of mat. multi.

$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times 1}$

$$C_{i1} = \sum_{k=1}^{n} A_{ik} B_{k1} \quad , \quad 1 \le i \le m$$



# Matrix powers

$$A^k := \underbrace{A \cdot A \cdots A}_{k}, \quad A \in \underbrace{\mathbb{R}^{n\times n}}_{\text{square mat.}}$$

Ex. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad A^3 = ?$

$$A^2 = AA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A \cdot A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A^2 A = A \cdot A^2$$

<u>Fact</u> $A^k$ is well defined.

<u>Ex</u>. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\neq$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

In general, order matters!

non-commutative.

# Identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & \cdots & & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

# Matrix transpose

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$A^T = \begin{bmatrix} A_{11} & \cdots & A_{m1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

$$(A^T)_{ji} = A_{ij}, \quad 1 \le i \le m$$
$$1 \le j \le n$$

Facts (i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $c \in \mathbb{R}$, $(cA)^T = c \cdot A^T$

(iv) $(AB)^T = B^T A^T$ ← verify dim. is consistent.

Pf: (iv) $A \in \mathbb{R}^{p \times m}$, $B \in \mathbb{R}^{m \times n} \Rightarrow AB \in \mathbb{R}^{p \times n}$

$(AB)^T \in \mathbb{R}^{n \times p}$, $B^T A^T \in \mathbb{R}^{n \times p}$.

$\forall \; 1 \leq i \leq n, \; 1 \leq j \leq p,$

$$[(AB)^T]_{ij} = [AB]_{ji}$$

$$= \sum_{k=1}^{m} A_{jk} B_{ki} .$$

$$[B^T A^T]_{ij} = \sum_{k=1}^{m} (B^T)_{ik} (A^T)_{kj}$$

$$= \sum_{k=1}^{m} B_{ki} \; A_{jk}$$

$$= \sum_{k=1}^{m} A_{jk} B_{ki} \quad \longleftarrow \text{Scalar multiplications commute!}$$

Therefore

$$(AB)^T = B^T A^T. \qquad \square.$$