Lec 7. Matrix inverse

$$3 \times \frac{1}{3} = 1.$$

$T : \mathbb{R} \to \mathbb{R}$ $\qquad$ $T^{-1} : \mathbb{R} \to \mathbb{R}$

$x \mapsto 3x.$ $\qquad$ $x \mapsto \frac{1}{3}x$

$$(T \circ T^{-1})(x) = 1 \cdot x = x.$$

$T : \mathbb{R} \to \mathbb{R}$ $\qquad$ T^{-1} does NOT exist.

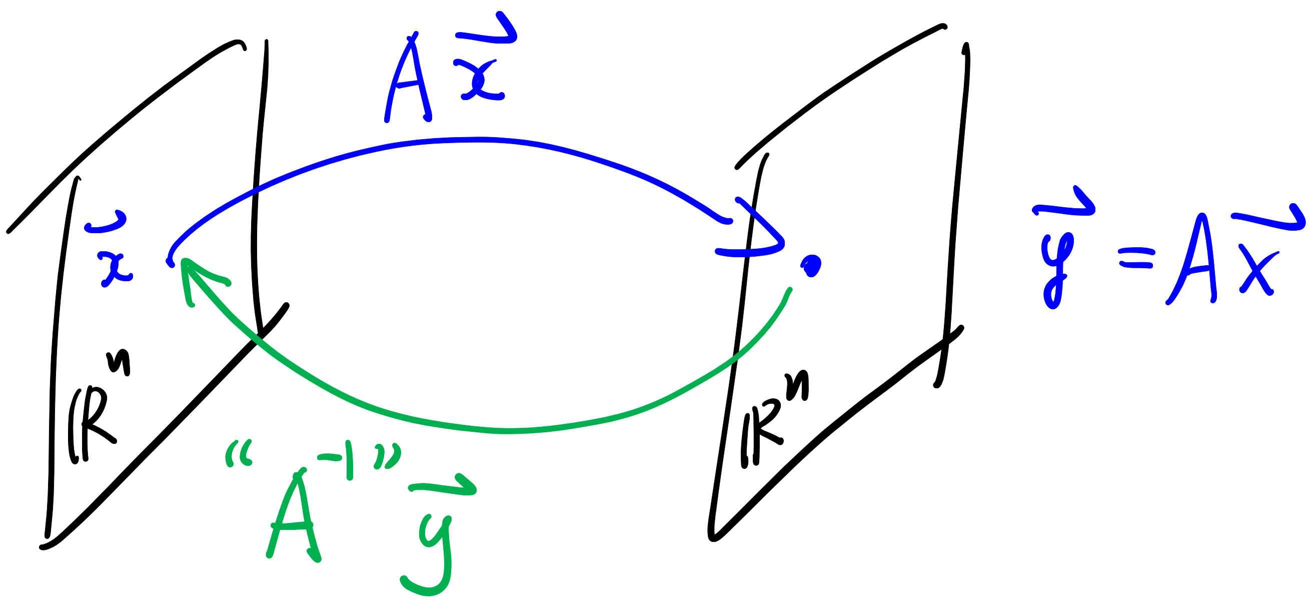
$x \mapsto 0 \cdot x = 0.$

$T: \mathbb{R}^n \to \mathbb{R}^n$ lin. trans.

$\vec{x} \mapsto A\vec{x}$. $A \in \mathbb{R}^{n \times n}$

Q: Can we define T^{-1}

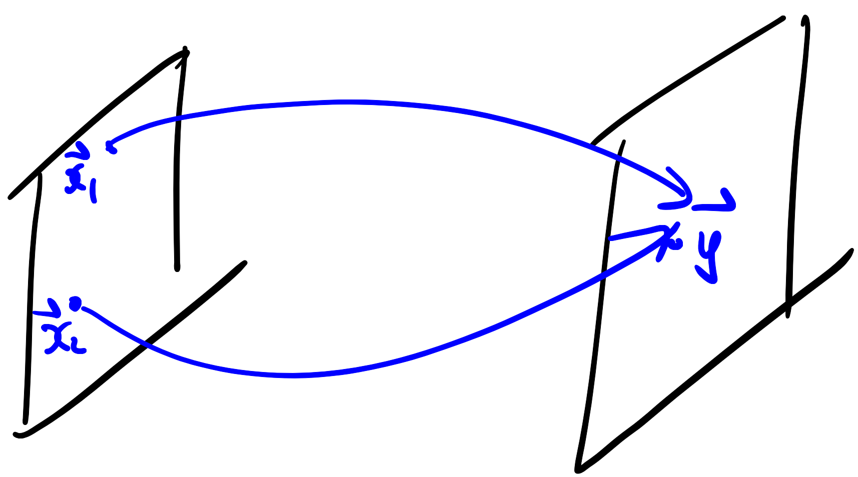
$(T \circ T^{-1})(\vec{x}) = \vec{x}$. $(T^{-1} \circ T)(\vec{x}) = \vec{x}$



① T is onto. $\mathrm{Image}(T) = \mathbb{R}^n$

$\{T\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$.

② T is one-to-one.



$$T \begin{cases} \text{onto} \\ \text{one-to-one} \end{cases} \Rightarrow \text{bijective}$$

$$T : x \mapsto Ax.$$

$$A \xrightarrow{\text{REF}} \begin{bmatrix} \blacksquare & * & * & * \\ & \blacksquare & * & * \\ & & \blacksquare & * \\ & & & \blacksquare \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I_n$$

Def $A \in \mathbb{R}^{n \times n}$ is invertible

if there exists $C \in \mathbb{R}^{n \times n}$ s.t.

$$AC = I_n \quad (\text{and} \quad CA = I_n)$$

C is called inverse of A. written as A^{-1}

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$C = [\vec{c}_1 \ \vec{c}_2]$

$A[\vec{c}_1 \quad \vec{c}_2] = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Updownarrow$

$A\vec{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2 lin. sys.

$A\vec{c}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

exchange 2 row

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

RREF $= I_2$.

$C = A^{-1}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \in \mathbb{R}^{n \times n}$. inv. $\quad A\vec{x} = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$.

We already have $A^{-1} \in \mathbb{R}^{n \times n}$.

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$\parallel$$

$$(A^{-1}A)\vec{x} = \vec{x}$$

$$\Longrightarrow \vec{x} = A^{-1}\vec{b}$$

$A, B, C \in \mathbb{R}^{n \times n}$. A $\boxed{\text{invertible.}}$

$$AB = AC$$

$$\begin{array}{ccc} A^{-1}(AB) & = & A^{-1}(AC) \\ \parallel & & \parallel \\ (A^{-1}A)B & & (A^{-1}A)C \end{array}$$

$$\Rightarrow B = C.$$

Scalar. $a \neq 0$

$$ab = ac \Rightarrow b = c.$$

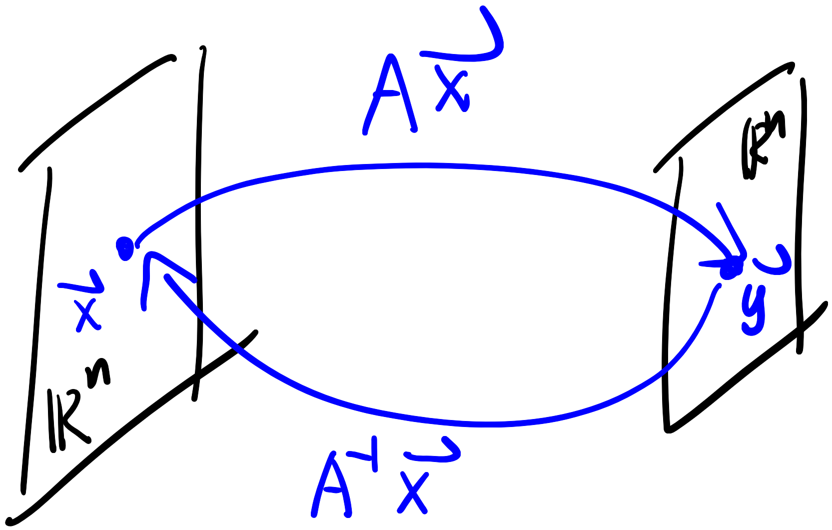
A invertible is KEY.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$AB = AC = 0 \not\Rightarrow B = C.$$

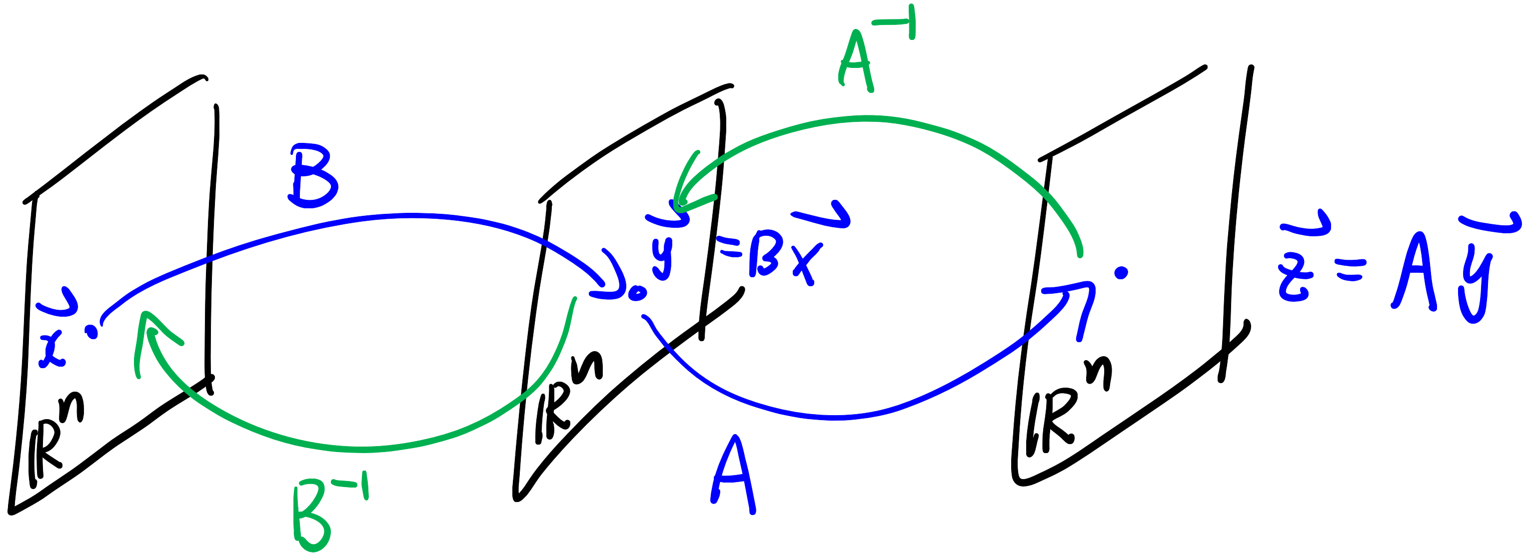
Thm. $A \in \mathbb{R}^{n\times n}$. inv.

$$(A^{-1})^{-1} = A.$$



<u>Thm</u>. $A, B \in \mathbb{R}^{n\times n}$ invertible.

$$(AB)^{-1} = B^{-1}A^{-1}$$



Pf: $(AB)\cdot(B^{-1}A^{-1})$

$$= A\,\underbrace{(BB^{-1})}_{I_n}\,A^{-1}$$

$$= AA^{-1} = I_n.$$

$$(B^{-1}A^{-1})(AB) = I_n \qquad \square.$$