Lec 9

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\text{Null}(A) = \left\{ x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, x_2 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

parametric vector form

$\dim \mathrm{Col}(A) = 2.$ $\qquad$ $\mathrm{rank}(A)$

$\dim \mathrm{Null}(A) = 1.$

$$\underset{\#\text{cols}}{3} = \underset{\substack{\dim \mathrm{Col}(A)\\ \#\text{pivots}}}{2} + \underset{\substack{\dim \mathrm{Null}(A)\\ \#\text{ free var.}}}{1}$$

Thm (Rank thm)

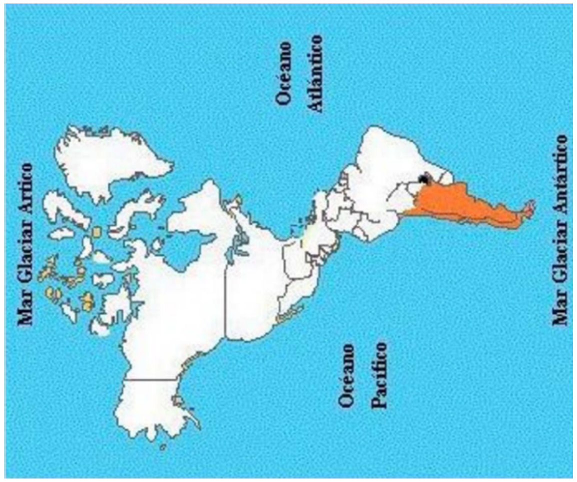
$A \in \mathbb{R}^{m \times n}$

$$n = \mathrm{rank}(A) + \dim \mathrm{Null}(A)$$

Vector space

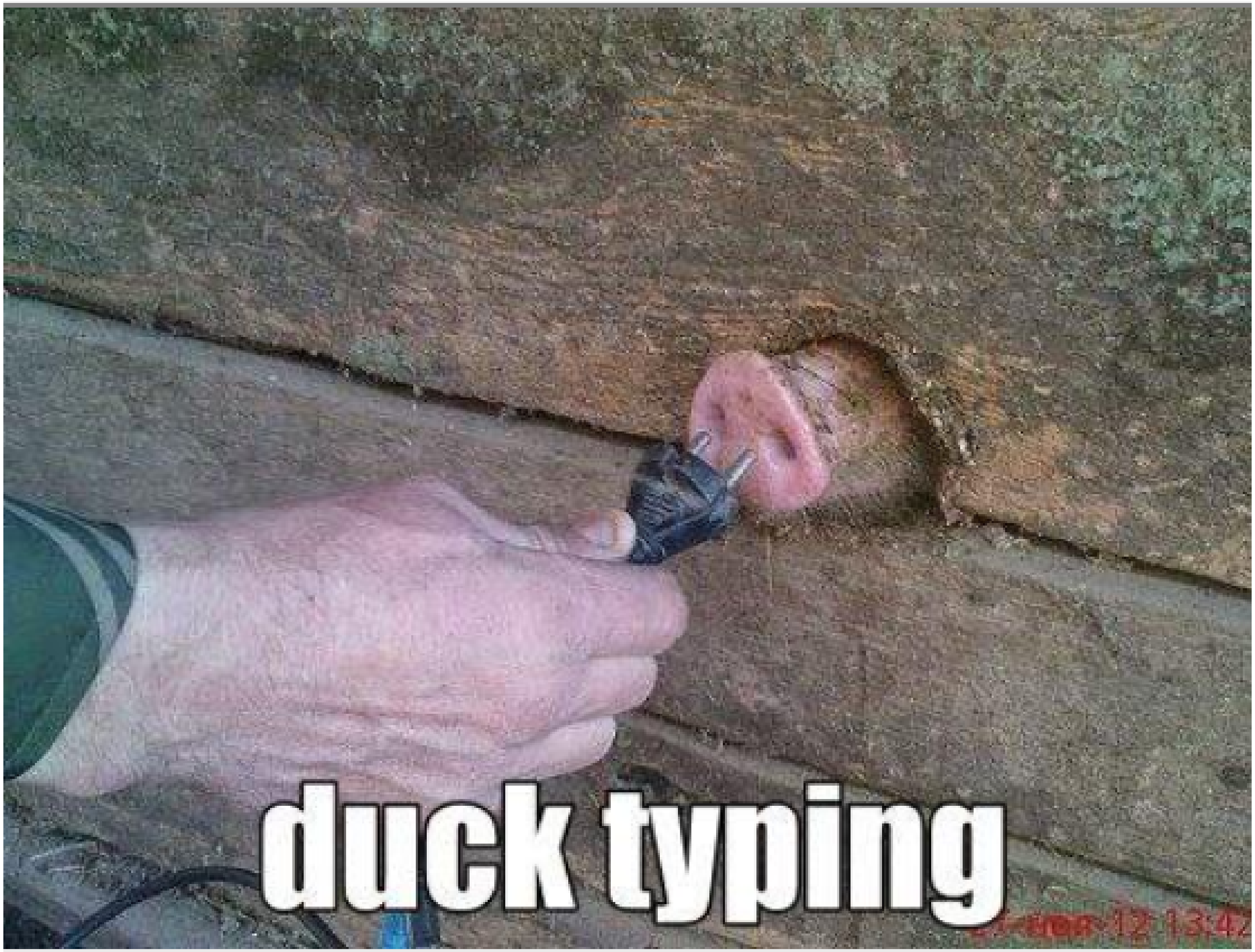
axiomatic approach

(1) looks



(2) quacks

$\Rightarrow$ duck.



A humorous and apt representation of duck typing. Source: Mastracci, 2014.*

A mathematical duck looks like...

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} + \mathbf{v}$, is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

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$\rightarrow \mathbb{R}$ or $\mathbb{C}$

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scalar 1.

Ex. $V = \mathbb{R}^n$

Ex. $V = \{ f : \mathbb{R} \to \mathbb{R}$ is a functions $\}$.

addition : $(f+g)(x) = f(x) + g(x)$. $\forall x \in \mathbb{R}$

scalar mult : $(cf)(x) = c\, f(x)$. $\forall x \in \mathbb{R}$, $c \in \mathbb{R}$

$\boxed{\text{Ex}}$. $V = \mathbb{P} = \{$ polynomial functions of finite degree : $\mathbb{R} \to \mathbb{R} \}$.

$$p(x) = \underbrace{a_n}_{\neq 0} x^{n} + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

$\deg(p)$

$\boxed{\text{Ex.}}$ $V = \mathbb{P}_n = \{\text{polynomial} : \mathbb{R} \to \mathbb{R}$

$\text{of degree} \leq n\}$.

$\boxed{\text{Ex}}$ $V = \{$ polynomial : $\mathbb{R} \to \mathbb{R}$ of degree $= n \}$.

$p(x) = x^n$.

$0 \cdot p(x) = 0$ not closed under scalar mult.

$p(x) + (-p(x)) = 0$, not closed under addition.

Ex. $\mathbb{R}^1 \not\subseteq \mathbb{R}^2$

$\{x \mid x \in \mathbb{R}\}$ $\quad$ $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1, x_2 \in \mathbb{R} \right\}$.

Ex. $\mathbb{P}_1 \subseteq \mathbb{P}_2$ ✓

$\mathbb{P}_1 = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$

$\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

Ex. $V = S = \{(a_1, a_2, a_3, \cdots) \mid a_i \in \mathbb{R}, \forall i\}$.

infinite sequence.

addition: $(a_1, a_2, a_3, \cdots) + (b_1, b_2, b_3, \cdots)$

$$= (a_1+b_1, a_2+b_2, a_3+b_3, \cdots)$$

scalar mult $\quad c \cdot (a_1, a_2, \cdots) = (ca_1, ca_2, \cdots)$

zero vector $\quad (0, 0, \cdots)$

Ex. $V =$ collection of all possible ducks differing only in width

 $+$  $=$ 

3  $=$ 

Ex. $V = \{A \in \mathbb{R}^{n\times n} \text{ is of REF}\}$.

$$\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \end{bmatrix} + \begin{bmatrix} \boxed{2} & 4 \\ 0 & \boxed{3} \end{bmatrix} = \begin{bmatrix} \boxed{3} & 6 \\ 0 & \boxed{4} \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{1} \end{bmatrix} + \begin{bmatrix} \boxed{-1} & 2 \\ 0 & \boxed{1} \end{bmatrix} = \begin{bmatrix} 0 & \boxed{4} \\ 0 & \boxed{2} \end{bmatrix} \quad \times$$

Not closed under addition.